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Aspects of String Theory Compactifications

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Aspects of String Theory Compactifications

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To my parents.

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Aspects of String Theory Compactifications

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String Theory is defined consistently when the dimensionality of the spacetime is ten. To make contact with the apparent four-dimensional world that we live in, we need to "compactify" six of the dimensions of String Theory. This dissertation is dedicated to the study of various physical and mathematical aspects of this problem.

In the first part of this dissertation, we construct the mirror of the Beauville manifold which is a Calabi-Yau 3-fold with non-abelian fundamental group. (The preservation of supersymmetry dictates that the internal space into which String Theory is compactified is a so-called Calabi-Yau manifold.) We use the conjecture of Batyrev and Borisov to find the previously misidentified mirror of its universal covering space, $\mathbb{P}^7[2,2,2,2]$. The monomial-divisor mirror map is essential in identifying how the fundamental group of the Beauville manifold acts on the mirror of $\mathbb{P}^7[2,2,2,2]$. Once we find the mirror of the Beauville manifold, we confirm the existence of the threshold bound

state around the conifold point, which was originally conjectured in [12]. We also consider how the quantum symmetry group acts on the D-branes that become massless at the conifold point and show the action proposed in [11] is compatible with mirror symmetry.

In the second part, we discuss an important subclass of D-branes on a Calabi-Yau manifold, X, which are in 1-1 correspondence with objects in $D(X)$, the derived category of coherent sheaves on X. We study the action of the monodromies in Kähler moduli space on these D-branes. We refine and extend a conjecture of Kontsevich about the form of one of the generators of these monodromies (the monodromy about the "conifold" locus) and show that one can do quite explicit calculations of the monodromy action in many examples. As one application, we verify a prediction of Mayr about the action of the monodromy about the Landau-Ginsburg locus of the quintic. Prompted by the result of this calculation, we propose a modification of the derived category which implements the physical requirement that the shift-by-6 functor should be the identity.

The last part of the dissertation is devoted to an F-theory compactification. We consider F-theory on an elliptically fibered Calabi-Yau 4-fold. We review the mechanisms used to stabilize various moduli in the theory. Especially, we take a closer look at Kähler moduli stabilization by the generation of non-perturbative superpotentials and argue that stabilization of all Kḧaler moduli in this way is non-generic. We consider an example where explicit analytic computation is possible and show that in this example, when all Kähler moduli are stabilized, the overall size is big enough for the supergravity approximation used here to be valid.

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Chapter 1

Introduction

Quantum Field Theory is one of the most successful theories of nature. The Standard Model of particle physics is a Quantum Field Theory, and it does a marvellous job of explaining three of the four known interactions in nature, the exception being gravity. Encouraged by this extraordinary success, there have been numerous attempts to quantize gravity in the Quantum Field Theory framework. However, obstinate problems have hindered all these attempts, among them non-renormalizability of the theory obtained by naively quantizing the Einstein-Hilbert action. This strongly suggests that a quantum theory of gravity might take a very different form from a conventional Quantum Field Theory.

String Theory is the leading candidate for a mathematically consistent quantum theory of gravity. String Theory, as it exists now, does not have a complete definition. Only a perturbative background dependent definition of String Theory is known. Perturbative String Theory is consistent only in 10 dimensional space-time. To make a realistic model from String Theory, one needs to make a connection with the apparently 4 dimensional world that we live in. There have been two major approaches to this problem. One is to "compactify" String Theory on a 6 dimensional internal manifold to obtain a 4 dimensional low energy effective theory[13]. That is, we assume that the vacuum of String Theory is of the form $M_4 \times X$, where M_4 is 4 dimensional Minkowski space and X is some compact six dimensional manifold. At low energies, one cannot excite modes moving along the compact directions rendering the low energy effective theory 4 dimensional. Many properties of the low energy effective theory are determined by the topology of X. For example, the existence of unbroken supersymmetry requires X to be a Calabi-Yau 3-fold. This strong connection between physics at low energy scale and the topology of X has made String Theory an exciting topic both to physicists and mathematicians.

The other approach to the extra dimension problem is the so-called brane world scenario[16, 48]. Here, we assume that part or all of the observable matter is confined to a 4 dimensional sub-manifold of the higher dimensional space-time. A concrete example of such situations is open string modes on a D3 brane of Type IIB String Theory. Since open strings cannot move off from the D3 brane, they see the 4 dimensional worldvolume of the D3 brane as their space-time. However, the D3 brane does not bind closed string modes such as the graviton. One way of overcoming this is to place other branes into the background, inducing an exponential warp factor. This warp factor ensures that the effective volume of the extra dimesions finite. Actually, one can employ both approaches, together. In this case, one compactifies String Theory on an internal manifold and also includes D branes in the background.

String Theory compactification, together with supersymmetry, also opens up windows to study non-perturbative properties of String Theory. As we mentioned earlier, String Theory does not have a complete definition, but it is still possible to study its various non-perturbative effects. Supersymmetry and dualities have been indispensable tools in exploring regimes inaccessible to perturbative String Theory. Among the many dualities discovered in String Theory compactifications, mirror symmetry is one of the most studied. It is a duality between Type IIA string theory compactified on one Calabi-Yau 3-fold and Type IIB on another. This second Calabi-Yau is called the mirror of the first[45]. Since it was first proposed in [34], it has been studied extensively. Many examples of Calabi-Yau 3-folds and their mirrors have been constructed [8, 14, 15].

In the first part of this dissertation, we construct the mirror of the Beauville manifold. The Beauville manifold[7] is the only known example of a Calabi-Yau 3-fold with non-abelian fundamental group. it has the fundamental group Q, the group of unit quaternions. There are several reasons that Calabi-Yau manifolds with non-abelian fundamental groups are interesting. For one, they may be used to construct phenomenologically realistic models. When heterotic string theory is compactified on a Calabi-Yau manifold, the gauge group is broken down to the subgroup that is preserved by the vacuum expectation values of the gauge fields. The vacuum expectation values should be chosen carefully to preserve supersymmetry and to ensure the anomaly cancelation. Most of the times, the gauge group so obtained is too big to be phenomenologically interesting. One can break the gauge group further by turning on Wilson lines. However, if the fundamental group of the Calabi-Yau manifold is abelian, turning on Wilson lines does not change the rank of the gauge group. Hence, if we want to reduce the rank of the gauge group, a non-abelian fundamental group is necessary [31].

Also, Calabi-Yau manifolds with non-abelian fundamental groups provide good examples for studying various aspects of D-branes. For instance, in [12], it is conjectured that, on the level of K-theory, the monodromy about the conifold locus (principal component of the discriminant locus) is of the form

$$v \to v - \sum_R (v, W_R) W_R. \tag{1.1}$$

The sum is over all irreducible representations of the fundamental group G, and W_R is the flat bundle built using the irreducible representation R. If G is non-abelian, this implies there are threshold bound states of D6-branes that are stable and become massless at the conifold locus. Since the existence of these states are guaranteed by neither the BPS condition nor by K-theory, it is interesting to study them.

Another interesting application is the action of the quantum symmetry group on D-branes. Any non-simply connected Calabi-Yau manifold, X can be written as $X = Y/G$. Y is the universal covering space and G is the fundamental group of X. Hence, string theory on X is described by an orbifold conformal field theory and this orbifold CFT has a quantum symmetry group, $G/[G, G]$. The action of this group on states in the closed string sector is known and well understood. It is interesting to see how it acts on non-perturbative states like D-branes. Its action on D-brane charges in K-theory has been conjectured in [11]. Since it applies to both A-branes and B-branes, it is interesting to check whether the conjectured action is compatible with mirror symmetry. If the fundamental group is non-abelian, it becomes more interesting since the quantum group is not any longer isomorphic to the fundamental group.

We construct the mirror of the Beauville manifold by taking the quotient of the mirror of its universal covering space, $\mathbb{P}^7[2,2,2,2]$ by Q. The mirror of $\mathbb{P}^7[2,2,2,2]$ has been considered in the literature [8]. However, as we will explain later, the conjectured mirror has a significant flaw. In this paper, we construct its correct mirror using the Batyrev-Borisov conjecture [6, 10]. The monomial-divisor mirror map [1] is the essential tool in finding the Q-action on the mirror. To do so, we extend the original argument a little bit since $\mathbb{P}^7[2,2,2,2]$ is a complete intersection Calabi-Yau manifold.

The procedure we use here can be applied to any non-simply connected Calabi-Yau manifold X. One finds the mirror of the universal covering space Y, and applies the monomial-divisor mirror map to find how the fundamental group G, acts on the mirror. There is a subtlety though. The monomial-divisor mirror map is valid only in the large radius limit. There is no guarantee that Y's mirror is in the neighborhood of the large radius limit. Also, the definition of monomials depends on the choice of homogeneous coordinates. Different choices of homogeneous coordinates will give different monomial-divisor mirror maps. Therefore, we need to choose the homogeneous coordinates carefully.

In the second part of the dissertation, we consider D-branes of Type II String Theory compactified on a Calabi-Yau 3-fold X. A class of D-branes in this backgound are conjectured to be in 1-1 correspondence with objects in $D(X)$, the derived category of coherent sheaves on X [23–25]. It is certainly not true that all, or even most D-branes on X can be described this way. The ones which *can* are the B-type branes which are related to D-branes in the B-type topological string theory on X. These form a nice subclass of B-type branes which is, moreover, carried into itself by the action of the monodromies. As proposed by [41] and elaborated upon by [37, 38] and [50, 51], the monodromies act on these D-branes by *auto-equivalences* of the derived category.

We will review Kontsevich's formula of the monodromy action on the derived category and will propose two modifications of his proposal. One will correct the grading, so that the D-branes which become massless at the (mirror of the) conifold point are invariant under the conifold monodromy (as we expect to be the case, since there is a local field theory description of the physics near the conifold if one introduces these D-branes as fundamental fields in the action). The second modification will be required to take account of the physics of *nonsimply-connected* Calabi-Yau manifolds. We will check these proposals by doing some explicit computations of D-brane monodromies on some simple Calabi-Yau manifolds. We will treat in detail the case of the quintic in $\mathbb{P}^4$ and the orbifold of the quintic by a freely-acting $\mathbb{Z}_5$ symmetry. We will find explicit formulæ for the monodromy action on wrapped D6-, D4-

D2- and D0-branes. Among other things, we will verify a prediction of Mayr [42] about the orbit of the D6-brane under monodromy about the Landau-Ginsburg point.

However, in doing this explicit computation, we encounter a surprise, namely that the 5^{th} power of the Landau-Ginsburg monodromy is not the identity in the derived category. Rather, it is equal to the shift-by-12 functor. In §3.3.3, we propose a modification of the derived category which implements the physical requirement that the shift-by-6 functor is an isomorphism. In this modified category, $M_{LG}^5 \simeq \mathbb{1}$. We propose this modified category as the correct category for B-type topological open strings. The new category has *more* isomorphisms, and hence *fewer* isomorphism classes (thus fewer D-branes). This resolves a puzzle [23] about the correspondence between the topological and the physical open string theory.

In the last part of the dissertation, we consider F-theory compactifications on a Calabi-Yau 4-fold. The interest in this arises partly because Kachru, Kallosh, Linde and Trivedi used this compactification, together with supersymmetry breaking by anti-D3 branes, to obtain meta-stable de Sitter vacua in String Theory[39]. Motivated by observations that strongly suggest a small positive cosmological constant[47, 49], a great deal of effort had been put in to try to construct de Sitter vacua in String/M Theory. However, no specific construction of true de Sitter vacua in String Theory were known, and theoretical arguments were even advanced that these should not exist[5, 26, 29]. Furthermore, understanding the quantum nature of gravity in an asymptotically de Sitter universe poses a lot of conceptual challenges[54].

KKLT circumvented these obstacles by suggesting that the universe that we live is not a true de Sitter vacuum but rather a meta-stable one. It eventually decays to 10 dimensional flat Minkowski space. However, it does so only through very slow tunneling processes and one can achieve an extremely long lifetime. At the same time, the lifetime is always shorter than the Poincare recurrence time, which has been used as an argument against eternal de Sitter space as a vacuum[29].

KKLT's construction of meta-stable de Sitter vacua consists of two steps. In the first step, they considered F-theory compactification on a Calabi-Yau 4-fold with a flux turned on. The flux generates a superpotential that freezes the complex structure moduli of the Calabi-Yau but leaving Kähler structure moduli free. At the leading order in α' and g_s, the low energy effective theory has a no-scale structure that does not fix the overall size of the Calabi-Yau. To stabilize the Kähler moduli, they considered non-perturbative effects that ruin the no-scale structure. Then they argued that the corrections to the superpotential due to the non-perturbative effects stabilize all Kähler moduli leading to a supersymmetric anti-de Sitter vacuum.

In the second step, they broke the supersymmetry by adding an anti-D3 brane into the background. They argued that with a careful choice of the flux, exponentially large warping can be achieved so that the inclusion of the anti-D3 brane does not disturb the system much. The presence of the anti-D3 brane lifts the vacuum and one ends up with a positive cosmological constant without destabilizing the minimum. The result is a meta-stable de Sitter vacuum.

In the dissertation, we consider F-theory on an elliptically fibered Calabi-Yau 4-fold and review the mechanisms used to stabilize various moduli in the theory. Especially, we take a closer look at Kähler moduli stabilization by the generation of non-perturbative superpotentials and argue that stabilization of all Kḧaler moduli in this way is non-generic. We consider an example where explicit analytic computation is possible and show that in this example, when all Kähler moduli are stabilized, the overall size is big enough for the supergravity approximation used here to be valid.

The first two parts of this thesis are mainly based on papers [21, 46] and the last part is unpublished work.

Chapter 2

Finding the Mirror of the Beauville Manifold

In this chapter, we construct the mirror of the Beauville manifold. First, we find the mirror of its universal covering space, $\mathbb{P}^7[2,2,2,2]$ using the Batyrev-Borisov conjecture [6, 10]. We use the monomial divisor map [1] to identify the action of the fundamental group Q on the mirror of $\mathbb{P}^7[2,2,2,2]$. Doing so, we extend the original argument in [1] a little bit so that it can be applied to a complete intersection Calabi-Yau. Then, the mirror of the Beauville manifold is constructed by taking the quotient of $\mathbb{P}^7[2,2,2,2]$'s mirror by Q.

2.1 The Beauville Manifold

There is only one known example of Calabi-Yau manifolds with non-abelian fundamental groups. It is the manifold Beauville constructed in [7]. The fundamental group of this Calabi-Yau manifold is the group of unit quaternions:

$$Q = \{\pm 1, \pm I, \pm J, \pm K\} \tag{2.1}$$

with multiplication law

$$
\begin{aligned}
&IJ = K \qquad \text{(and cyclic)} \\
&I^2 = J^2 = K^2 = -1
\end{aligned}
\tag{2.2}
$$

Before we review the construction of this manifold, let us recall some facts about the group theory of Q [12]. First, there is an exact sequence,

$$
0 \to \mathbb{Z}_2 \to Q \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 0 \tag{2.3}
$$

where the commutator subgroup of Q is the $\mathbb{Z}_2$ subgroup, $\{1, -1\}$ and its abelianization, $Q/[Q,Q] = \mathbb{Z}_2 \times \mathbb{Z}_2$.

The irreducible representations of Q are as follows. There are four 1-dimensional irreps: the trivial rep V_1 and the representations V_I, V_J, and V_K. In V_I, ± 1 and $\pm I$ are represented by 1 while $\pm J$ and $\pm K$ are represented by -1 (and similarly for $V_{J,K}$). There is also a 2-dimensional representation, V_2. $\pm I, \pm J$ and $\pm K$ act on V_2 by $\pm i\sigma_3, \pm i\sigma_2$ and $\pm i\sigma_1$. The representation ring is

$$
\begin{aligned}
&V_2 \otimes V_2 = V_1 \oplus V_I \oplus V_J \oplus V_K \\
&V_\alpha \otimes V_2 = V_2 \qquad \alpha = 1, I, J, K \\
&V_I \otimes V_J = V_K \qquad \text{(and cyclic)}
\end{aligned}
\tag{2.4}
$$

The group homology of Q is

$$
\mathrm{H}_1(Q) = Q/[Q,Q] = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \qquad \mathrm{H}_2(Q) = 0 \tag{2.5}
$$

Now, let's construct the Beauville manifold. Let V_8 be the regular representation of Q and $\mathbb{P}[V_8]$ be its projective space. The Q-action on V_8

induces a Q-action on $\mathbb{P}[V_8]$ and also on the space of quadrics in it. We choose 4 quadrics, one from each 1-dimensional irreducible representation. Then, the intersection Y, of these quadrics will be invariant under Q. What Beauville showed in [7] is that for generic enough choices, Y is a smooth Calabi-Yau manifold and Q acts freely on Y. Hence, we can take the quotient $X = Y/Q$ and X is a smooth Calabi-Yau manifold with fundamental group Q.

The Hodge numbers of Y and X are $h^{1,1}(Y) = 1$, $h^{2,1}(Y) = 65$ and $h^{1,1}(X) = 1$, $h^{2,1}(X) = 9$, respectively. It is easy to show that Q acts trivially on $\mathrm{H}_2(Y)$. Applying the Cartan-Leray Spectral Sequence as in [11], we can show that

$$\pi_* : \mathrm{H}_2(Y) \to \mathrm{H}_2(X) \tag{2.6}$$

is an isomorphism. Here, π_* is the push-forward of the projection $\pi : Y \to X$. The Poincaré duality implies that the pull-back

$$\pi^* : \mathrm{H}^2(X) \to \mathrm{H}^2(Y) \tag{2.7}$$

is also an isomorphism. The classical consideration around the large radius limit shows that the Kähler moduli space of X is the same as that of Y [4].

2.2 Mirror of Y

In the literature [8], the mirror of $Y = \mathbb{P}^7[2,2,2,2]$ has been conjectured to be $Z = \mathbb{P}^7[2,2,2,2]/G$ where $G \simeq (\mathbb{Z}_4)^3$ is generated by[a]

$$\begin{aligned}
g_1 &: [X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8] \mapsto [\zeta X_1, \zeta^3 X_2, \zeta^2 X_3, X_4, X_5, X_6, X_7, X_8] \\
g_2 &: [X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8] \mapsto [X_1, X_2, \zeta X_3, \zeta^3 X_4, \zeta^2 X_5, X_6, X_7, X_8] \\
g_3 &: [X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8] \mapsto [X_1, X_2, X_3, X_4, \zeta X_5, \zeta^3 X_6, \zeta^2 X_7, X_8]
\end{aligned} \tag{2.8}$$

and ζ is a fourth root of unity. To make Z a Calabi-Yau manifold, we must choose 4 quadrics $G_1, \ldots, G_4$ such that they transform under G in the following way.

$$\begin{aligned}
g_1 &: [G_1, G_2, G_3, G_4] \mapsto [\zeta^2 G_1, G_2, G_3, G_4] \\
g_2 &: [G_1, G_2, G_3, G_4] \mapsto [G_1, \zeta^2 G_2, G_3, G_4] \\
g_3 &: [G_1, G_2, G_3, G_4] \mapsto [G_1, G_2, \zeta^2 G_3, G_4]
\end{aligned} \tag{2.9}$$

The most general such quadrics are (up to isomorphism):

$$\begin{aligned}
G_1 &= X_1^2 + X_2^2 - 2\psi X_3 X_4 \\
G_2 &= X_3^2 + X_4^2 - 2\psi X_5 X_6 \\
G_3 &= X_5^2 + X_6^2 - 2\psi X_7 X_8 \\
G_4 &= X_7^2 + X_8^2 - 2\psi X_1 X_2.
\end{aligned} \tag{2.10}$$

Z is singular. There are fixed points since G-action on Z is not free. Those singularities are expected since Z must have $h^{1,1} = 65$. However, there are worse singularities. For any value of ψ, Z contains points where the

[a] Actually, in [8], the group G was not explicitly given.

transversality of $G_1, \ldots, G_4$ fails. For example, at $[0,0,0,0,0,\sqrt{\psi}(1+i),i,1]$, $dG_1 = G_1 = G_2 = G_3 = G_4 = 0$. Since all those points are also fixed by some elements of G, one might try to resolve these singularities by blowing up.[b] Unfortunately, it does not work. Some orbifold singularities can be cured by blow-ups without changing the canonical class [33]. But, in our case, blowing up the singular locus will change the canonical class. That means that the resulting manifold is not any more a Calabi-Yau. Later, we will see how Z is related to the actual mirror.

To construct the mirror, we use the conjecture originally made by Borisov [10] and further developed by Batyrev and Borisov [6]. From now on, we will use toric geometry heavily. For the notation and review, see, for example, [9, 27, 33] and references therein.

2.2.1 Toric description of Y

Now, we describe $\mathbb{P}^7$ as a toric variety. Let N be a lattice of rank 7 and $e_1, \ldots, e_7$ be its generators. We define $e_8 = -e_1 - \ldots - e_7$ and also denote by $N_{\mathbb{R}}$ the real scalar extension of N. Let Σ be the fan in $N_{\mathbb{R}}$ whose cones are generated by proper subsets of the vectors $e_1, \ldots, e_8$. The toric variety associated to this fan is $\mathbb{P}^7$. Another way of realizing $\mathbb{P}^7$ as a toric variety is using a polyhedron. Consider the polyhedron $\Delta^* = \mathrm{Conv}\left(\{e_1, \ldots, e_8\}\right)$ and

[b]Mathematically, blowing up is a procedure for replacing a point with $\mathbb{P}^{n-1}$ where n is the dimension of the variety we are considering. In this paper, we generalize the notion to include replacing a subvariety of codimension more than 2 with a subvariety of codimension one which is not necessarily $\mathbb{P}^{n-1}$.

take its dual Δ in $M_{\mathbb{R}}$. Here, $M_{\mathbb{R}}$ is the real scalar extension of the dual lattice M. Then, $\mathbb{P}_{\Delta} = \mathbb{P}^7$.

To represent Y, the intersection of 4 quadrics, $G_1, \ldots, G_4$, we choose the following nef partition of vertices of Δ^*:

$$E_l = \{e_{2l-1}, e_{2l}\}, \qquad l = 1, \ldots, 4 \tag{2.11}$$

Choosing the nef partition amounts to specifying one monomial for each equation, G_l. To see this, we need some facts about divisors in a toric variety (see [27], section 3.3 for fuller explanation). First, recall that every toric variety has the torus action $T = N \otimes_{\mathbb{Z}} \mathbb{C}^*$. In study of toric varieties, we are mainly interested in T-stable divisors. For Weil divisors, we need codimension one irreducible subvarieties that are invariant under T-action. There are only finite number of such subvarieties and they are represented by one-dimensional cones, or edges, in the fan. Call them D_i. Then, T-stable Weil divisors are sums $\sum n_i D_i$ for integers n_i. Cartier divisors we are interested in are the ones that are mapped to themselves up to multiplicative constants under T so that their zeroes and poles are invariant. In the toric variety, they are described by continuous integral Σ-piecewise linear functions. Such a function ψ is defined by specifying $\psi|_{\sigma} \in M$ for each full dimensional cone σ in Σ. Of course, $\{\psi|_{\sigma}\}$ must satisfy the continuity condition:

$$(\psi|_{\sigma_1} - \psi|_{\sigma_2}) \perp \sigma_1 \cap \sigma_2 \tag{2.12}$$

Note that for every $m \in M$, there is a corresponding meromorphic function χ^m. Using this fact, we construct a Cartier divisor from $\{\psi|_{\sigma}\}$. Open sets are ones given by the full dimensional cones σ in the fan and local equations are $\chi^{-\psi|_\sigma}$. (Here, the $-$ sign is the convention widely used in the literature.) A Cartier divisor determines a Weil divisor. In our case, a Cartier divisor given by ψ will give Weil divisor $\sum -\psi(v_i)D_i$ where v_i is the first lattice point met along the edge representing D_i. Conversely, for a given Weil divisor $\sum n_i D_i$, we get a Cartier divisor if the toric variety is smooth. The representing function ψ of the Cartier divisor is uniquely determined by the condition $\psi(v_i) = -n_i$. A Cartier divisor D also determines a lattice convex polyhedron in $M_{\mathbb{R}}$ defined by

$$\Delta_D = \{y \in M_{\mathbb{R}} \mid \langle x, y\rangle \geq \psi(x) \quad \forall x \in N_{\mathbb{R}}\} \tag{2.13}$$

This polyhedron is called the support of global sections of $\mathcal{O}(D)$ since they (the global sections) are determined by the lattice points inside Δ_D:

$$\Gamma\left(X, \mathcal{O}(D)\right) = \bigoplus_{m \in \Delta_D \cap M} \mathbb{C} \cdot \chi^m. \tag{2.14}$$

Now, let's apply the above theory to our case. In Σ, there are 8 edges, τ_i, $i = 1, \ldots, 8$. Each τ_i is generated by vertex e_i of Δ^* and represents divisor $D_i = \{X_i = 0\}$. The nef partition above determines 4 Weil divisors:

$$W_l = D_{2l-1} + D_{2l} \tag{2.15}$$

These Weil divisors, in turn, determine Cartier divisors. Their representing functions $\psi_1, \ldots, \psi_4$ are given by

$$\psi_l|_{\sigma_i} = 2\omega_i - \omega_{2l-1} - \omega_{2l}. \tag{2.16}$$

Here, σ_i, $i = 1, \ldots, 8$ are full dimensional cones in Σ. Each σ_i is generated by $e_1, \ldots, \not{e}_i, \ldots, e_8$. Also, $\{\omega_1, \ldots, \omega_7\}$ is the dual basis of $\{e_1, \ldots, e_7\}$ and $\omega_8 = 0$. The partition being nef implies ψ_l's are convex functions. In terms of homogeneous coordinate, these divisors are:

$$X_{2l-1}X_{2l} \tag{2.17}$$

Therefore, E_l represents the monomial $X_{2l-1}X_{2l}$ in G_l.

From the nef partition, we define two sets of convex lattice polyhedra, $\Pi = \{\Delta_1, \ldots, \Delta_4\}$, $\Pi^* = \{\nabla_1, \ldots, \nabla_4\}$.

$$\begin{aligned} \Delta_l &= \{x \in M_{\mathbb{R}} \mid \langle x, y \rangle \geq \psi_l(y)\} \\ \nabla_l &= \text{Conv}\left(\{0\} \cup E_l\right) \end{aligned} \tag{2.18}$$

Δ_l is the support of global sections of $\mathcal{O}_{\mathbb{P}^7}(W_l)$. Using the convexity of ψ_l, one can show that

$$\Delta_l = \text{Conv}\left(\left\{\, \psi_l|_{\sigma_i} \right\}_{i=1,\ldots,8}\right). \tag{2.19}$$

Each Δ_l contains 36 lattice points:

$$\omega_i + \omega_j - \omega_{2l-1} - \omega_{2l}, \qquad i, j = 1, \ldots, 8 \tag{2.20}$$

representing the monomial X_iX_j in equation G_l.[c] Also, note that $0 \in \Delta_l$ and it represents the original monomial $X_{2l-1}X_{2l}$ specified by E_l. It is the only common point of $\Delta_1, \ldots, \Delta_4$.

[c]One way of seeing this is to consider the corresponding Weil divisor:

$$\sum_{k=1}^{8}(-\psi_l + \omega_i + \omega_j - \omega_{2l-1} - \omega_{2l})(e_k)D_k = D_i + D_j$$

2.2.2 Construction of mirror $\widehat{Y}$

We define a lattice polyhedron $\nabla^* = \mathrm{Conv}\,(\Delta_1 \cup \ldots \cup \Delta_4)$ and take its dual ∇. Let $\widehat{V}$ be the toric variety $\mathbb{P}_{\nabla}$. One can also describe $\widehat{V}$ using the fan $\widehat{\Sigma} = \mathcal{N}(\nabla)$, the normal fan of ∇. It is the union of the zero-dimensional cone $\{0\}$ together with the set of all cones

$$\sigma[\theta] = \mathbb{R}^{\geq 0} \cdot \theta \tag{2.21}$$

that are supporting the faces θ of ∇^*. $\widehat{\Sigma}$ has 32 edges, $\widehat{\tau}_{l,i}$, $l = 1, \ldots, 4, i = 1, \ldots, 8$, each generated by the vertex $\psi_l|_{\sigma_i}$ of ∇^*. In [10], Borisov showed that

$$\widehat{E}_l = \left\{ \psi_l|_{\sigma_i} \right\}_{i=1,\ldots,8} \tag{2.22}$$

is a nef partition of vertices of ∇^*. Let $\widehat{D}_{l,i}$ be the Weil divisor $\widehat{\tau}_{l,i}$ represents. As before, each $\widehat{E}_l$ induces a Weil divisor:

$$\widehat{W}_l = \sum_{i=1}^{8} \widehat{D}_{l,i}. \tag{2.23}$$

We choose 4 global sections, $\widehat{G}_l$, one from each $\mathcal{O}_{\widehat{V}}(\widehat{W}_l)$. Then, the mirror $\widehat{Y}$ of Y is a complete intersection of $\widehat{G}_1, \ldots, \widehat{G}_4$. Borisov also showed that the supporting polyhedron of global sections of $\mathcal{O}_{\widehat{V}}(\widehat{W}_l)$ is ∇_l. Lattice points in ∇_l will generate the global sections. Under the mirror symmetry, the roles of Π and Π^* are interchanged.

2.2.3 Geometry of $\widehat{Y}$

To understand the geometry of the mirror $\widehat{Y}$ better, we describe $\widehat{V}$ as a holomorphic quotient [17]. First, we introduce "homogeneous coordinates"

$\widehat{X}_{l,i}$, $l = 1, \ldots, 4$, $i = 1, \ldots, 8$. Each homogeneous coordinate $\widehat{X}_{l,i}$ is paired to a vertex $\psi_l|_{\sigma_i}$ of ∇^*. Now, $\widehat{V}$ can be written as[d]

$$\widehat{V} \simeq \frac{\mathbb{C}^{32} - F_{\widehat{\Sigma}}}{G} \tag{2.24}$$

The above expression needs some explanation. $\mathbb{C}^{32}$ is the space whose (plain) coordinate functions are $\widehat{X}_{l,i}$. $F_{\widehat{\Sigma}}$ is the subset of $\mathbb{C}^{32}$ determined by the fan. If we realize this via a gauge linear sigma model, then $F_{\widehat{\Sigma}}$ is the set of excluded points by the D-term conditions. The group G acts on $\mathbb{C}^{32}$ as follows:

$$\widehat{X}_{l,i} \mapsto \Lambda_{l,i}\widehat{X}_{l,i} \quad \text{(no sum)} \tag{2.25}$$

where $\Lambda_{l,i}$'s are non-zero complex numbers satisfying:

$$\prod_{l,i} \Lambda_{l,i}^{\langle n, \psi_l|_{\sigma_i}\rangle} = 1 \tag{2.26}$$

for any $n \in N$. This group is the complexified gauge group of the gauge linear sigma model. Its phase part is the actual gauge group and the magnitude part is fixed by D-term [52]. Plugging (2.16), we get 7 independent equations:

$$\begin{aligned} \prod_{j=1}^{7} \Lambda_{m,j} &= \prod_{l=1}^{4} \Lambda_{l,2m-1}^2 \\ \prod_{j=1}^{7} \Lambda_{m,j} &= \prod_{l=1}^{4} \Lambda_{l,2m}^2 \end{aligned} \tag{2.27}$$

[d]Strictly speaking, it only makes sense as a categorical quotient since $\widehat{\Sigma}$ is not simplicial [17]. Here, we are actually considering the family of Calabi-Yau manifolds and $\widehat{V}$ is a special point on the Kähler moduli space. For generic points on the moduli space, the quotient above makes sense as a geometric one. Even at those points corresponding to non-simplicial fans, the mirror would be given by the geometric quotient rather than one given by toric geometry since it will be realized via a gauge linear sigma model.

for $m = 1, \ldots, 4$. One of the above equations follows from the rest. It turns out that G is isomorphic to $(\mathbb{C}^*)^{25} \otimes (\mathbb{Z}_2)^3$. The discrete subgroup, $G_{torsion} \simeq (\mathbb{Z}_2)^3$, of G non-trivially acts on the homogeneous coordinates. Because of this, $\widehat{X}_{l,i}$'s are not global sections. Only monomials that are invariant under $G_{torsion}$ will be global sections or Cartier divisors. However, the zeroes of $\widehat{X}_{l,i}$ is a globally well defined set. It is shown in [17]

$$\widehat{D}_{l,i} = \left\{ \widehat{X}_{l,i} = 0 \right\}. \tag{2.28}$$

Now, we use the homogeneous coordinates to express $\widehat{G}_l$. The nef partition $\widehat{E}_l$ gives a monomial for each $\widehat{G}_l$. They are

$$\prod_{i=1}^{8} \widehat{X}_{l,i} \tag{2.29}$$

Other terms in $\widehat{G}_l$ are determined by the lattice points in ∇_l as explained above. ∇_l has three such points $0, e_{2l-1}$, and e_{2l}. The corresponding monomials are:

$$\begin{aligned} e_{2l-1} &: \prod_{m=1}^{4} \widehat{X}^2_{m,2l-1} \\ e_{2l} &: \prod_{m=1}^{4} \widehat{X}^2_{m,2l} \end{aligned} \tag{2.30}$$

Then, the most general $\widehat{G}_1, \ldots, \widehat{G}_4$ are (up to isomorphism)

$$\begin{aligned}
\widehat{G}_1 &= \prod_{m=1}^{4} \widehat{X}_{m,1}^2 + \prod_{m=1}^{4} \widehat{X}_{m,2}^2 - 2\psi \prod_{i=1}^{8} \widehat{X}_{1,i} \\
\widehat{G}_2 &= \prod_{m=1}^{4} \widehat{X}_{m,3}^2 + \prod_{m=1}^{4} \widehat{X}_{m,4}^2 - 2\psi \prod_{i=1}^{8} \widehat{X}_{2,i} \\
\widehat{G}_3 &= \prod_{m=1}^{4} \widehat{X}_{m,5}^2 + \prod_{m=1}^{4} \widehat{X}_{m,6}^2 - 2\psi \prod_{i=1}^{8} \widehat{X}_{3,i} \\
\widehat{G}_4 &= \prod_{m=1}^{4} \widehat{X}_{m,7}^2 + \prod_{m=1}^{4} \widehat{X}_{m,8}^2 - 2\psi \prod_{i=1}^{8} \widehat{X}_{4,i}
\end{aligned} \tag{2.31}$$

$\widehat{V}$ and $\widehat{Y}$ are not smooth. Both have orbifold singularities. To see this, let's consider M', the sub-lattice of M that is generated by the vertices $\left\{ \psi_l|_{\sigma_i} \right\}$ of ∇^*. The fan $\widehat{\Sigma}$ is also a fan in M' since every cone in $\widehat{\Sigma}$ is a rational cone in M'. Therefore, one can construct a toric variety with lattice M' and fan $\widehat{\Sigma}$. Let's call it V'. In [27], it is shown

$$V' = \widehat{V} / \left(M/M' \right). \tag{2.32}$$

In our case, $M/M' \simeq (\mathbb{Z}_2)^3$. It is not a coincidence that M/M' and $G_{torsion}$ have the same form. The actual origin of $G_{torsion}$ is M/M'. V' has the same homogeneous coordinates $\widehat{X}_{l,i}$ as $\widehat{V}$ and can be written as a holomorphic quotient:

$$V' \simeq \frac{\mathbb{C}^{32} - F_{\widehat{\Sigma}}}{G'} \tag{2.33}$$

Group G' is defined similar to G. The only difference is that now n in (2.26) runs over N', the dual lattice of M'. G' is isomorphic to $(\mathbb{C}^*)^{25}$.

Let $\widehat{Y}'$ be the intersection of $\widehat{G}_1, \ldots, \widehat{G}_4$ in V'. From (2.32)

$$\widehat{Y}' = \widehat{Y} / \left(M/M' \right). \tag{2.34}$$

V' and $\widehat{Y}'$ are still singular since $\widehat{\Sigma}$ is not simplicial. However, the singularity of this type can be easily cured by subdividing non-simplicial cones in $\widehat{\Sigma}$. In the gauge linear sigma model point of view, we change Fayet-Iliopoulos parameters to more generic values. Even after subdivision, V' has still orbifold singularities. Fortunately, $\widehat{Y}'$ misses these singular points. As we will see later, the transversality of $\widehat{G}_1, \ldots, \widehat{G}_4$ holds for generic values of ψ. Hence, after subdivision, $\widehat{Y}'$ becomes smooth. That implies, for generic points in the moduli space, that all singularities of the mirror are orbifold singularities coming from the fixed points of $(\mathbb{Z}_2)^3$.

2.2.4 Z out of $\widehat{Y}$

In this sub-section, we consider the relation between the actual mirror $\widehat{Y}$ and the previously conjectured mirror Z. There is a way to get $\mathbb{P}^7/(\mathbb{Z}_4)^3$, the ambient space of Z, out of $\widehat{V}$. Choose 8 edges, τ_i', from $\widehat{\Sigma}$ as follows:

$$\begin{array}{cccc} \tau_1' = \widehat{\tau}_{4,1} & \tau_2' = \widehat{\tau}_{4,2} & \tau_3' = \widehat{\tau}_{1,3} & \tau_4' = \widehat{\tau}_{1,4} \\ \tau_5' = \widehat{\tau}_{2,5} & \tau_6' = \widehat{\tau}_{2,6} & \tau_7' = \widehat{\tau}_{3,7} & \tau_8' = \widehat{\tau}_{3,8} \end{array} \tag{2.35}$$

Let Σ' be the fan whose edges are τ_i''s. The toric variety associated to this fan is $\mathbb{P}^7/(\mathbb{Z}_4)^3$. It is possible to subdivide non-simplicial cones in $\widehat{\Sigma}$ in such a way that every subdivided cone is included in a cone in Σ'. Then, there is a proper map from the toric variety associated to this subdivision to $\mathbb{P}^7/(\mathbb{Z}_4)^3$ [27]. This map is the blow-up of $\mathbb{P}^7/(\mathbb{Z}_4)$ along the singular locus. Furthermore, we get the 4 equations in (2.10) defining Z from the sets, Π and Π^*.

From this construction, one might wonder if Z corresponds to a point in some corner of the Kähler moduli space of $\widehat{Y}$. The answer is "no". First of all, $\widehat{Y}$ is not a blow-up of Z. In $\widehat{V}$, there are natural liftings of the equation $G_1, \ldots, G_4$ in (2.10). It turns out that they are not $\widehat{G}_1, \ldots, \widehat{G}_4$, but $\widehat{G}_l$ times some monomials. Each monomial that multiplies $\widehat{G}_l$ involves only the homogeneous coordinates that corresponds to the exceptional divisors. These divisors are represented by edges of $\widehat{\Sigma}$ that are not edges of Σ'. The 4 equations obtained this way define a singular variety. This variety is reducible and contains $\widehat{Y}$ as an irreducible component. Only when we shrink the exceptional divisors to size zero, it becomes irreducible. However, it still contains singularities. This is the origin of Z's singularities where the transversality fails.

The image of $\widehat{Y}$ under the proper map is Z. Hence, it is interesting to see what happens to $\widehat{Y}$ when we "blow down" the exceptional divisors. The gauge linear sigma model describing Z has non-compact vacua. Recall this model has fields P_l that multiply G_l to comprise the superpotential. The theory has the flat direction along P_l's at points on Z where the transversality of G_l fails. However, this is not what happens in the gauge linear sigma model of $\widehat{Y}$. The flat directions will be replaced by compact manifolds parameterized by P_l's and the fields corresponding to the exceptional divisors.

To illustrate this point, let's consider a toy model. Take $\mathbb{P}^4$ and blow up a point $[0, 0, 0, 0, 1]$. The gauge linear sigma model describing this blown up space has 6 fields, $X_1, \ldots, X_5, T$ and two $U(1)$'s. The charges of the fields are:

$$\begin{array}{c|cccccc} & X_1 & X_2 & X_3 & X_4 & X_5 & T \\ \hline U(1)_1 & 1 & 1 & 1 & 1 & & -1 \\ U(1)_2 & & & & & 1 & 1 \end{array} \tag{2.36}$$

To have a Calabi-Yau hypersurface, the defining equation G must have charge $(3, 2)$ and is in the following form:

$$G(X_1, \ldots, X_5, T) = G_3(X_1, \ldots, X_4)X_5^2 + G_4(X_1, \ldots, X_4)X_5 T + G_5(X_1, \ldots, X_4)T^2 \tag{2.37}$$

where G_3, G_4, and G_5 are homogeneous polynomials of $X_1, \ldots, X_4$ with degree 3, 4, and 5, respectively. For generic enough G_3, G_4 and G_5, the hypersurface is a smooth Calabi-Yau manifold. At the point $X_1 = \ldots = X_4 = 0$, $G = dG = 0$. However, this point is excluded by D-term conditions:

$$\begin{aligned} r_1 &= |X_1|^2 + \ldots + |X_4|^2 - |T|^2 - 3|P|^2 \\ r_2 &= |X_5|^2 + |T|^2 - 2|P|^2 \end{aligned} \tag{2.38}$$

where P is a field introduced for G and, r_1 and r_2 are Fayet-Iliopoulos parameters. To represent the hypersurface in the blown up space, r_1 and r_2 must be taken positive. Now, let's blow down and see what happens. Blowing down corresponds to changing the value of r_1 to a negative number while keeping $r_1 + r_2$ positive. If there were no P in the first equation of (2.38), we would obtain $\mathbb{P}^4$. The first equation would determine the non-zero absolute value of T and the phase of T would be gauged away by $U(1)_1$. Since the point $X_1 = \ldots = X_4 = 0$ is allowed now, G becomes singular. That would allow P to take any value making the space of vacua non-compact. This is very similar to our situation. However, the presence of P in the first equation of (2.38) makes the difference. Even though G is singular, the space of vacua remains compact since the absolute value of P is bounded by the first equation in (2.38). Now, at the point $X_1 = \ldots X_4 = 0$, there is $\mathbb{W}^1_{(1,3)}$ parameterized by T and P. So far, our discussion has been classical. Of course, the quantum correction will change the shape of the vacua. However, we expect the quantum correction to be not so dramatic that the space of vacua remains compact.

2.3 Monomial-divisor mirror map

In section 2.1, we saw Q acts on the space of quadrics in $\mathbb{P}^7$. Hence, there is an induced Q-action on the complex structure moduli space, $\mathcal{C}_Y$, of Y. To define the Beauville manifold, we demand the complex structure to be in subset $\mathcal{C}^0_Y$ of $\mathcal{C}_Y$ that corresponds to choices of 4 quadrics, one from each irreducible representation of Q. With this choice of the complex structure, Q acts freely on Y as a subgroup of the automorphisms. On the mirror side, we expect there is the mirror Q-action on the Kähler moduli space $\mathcal{K}_{\widehat{Y}}$ of $\widehat{Y}$. To define the mirror of the Beauville manifold, we need to tune $\widehat{Y}$'s Kähler parameters so that the Kähler class is fixed by Q. Then, we hope to find a compatible Q-action on $\widehat{Y}$ itself and take the quotient of $\widehat{Y}$ by the Q-action.

To find the Q-action on $\mathcal{K}_{\widehat{Y}}$, we, presumably, need to solve the mirror map $\mu : \mathcal{C}_Y \to \mathcal{K}_{\widehat{Y}}$. However, with 65 parameters, it is practically impossible. (Here, we are talking about the mirror map between the complex structure of Y and the Kähler class of $\widehat{Y}$.) Instead of solving the mirror map directly, we will use the monomial-divisor mirror map [1]. It is the differential of the mirror map and is valid only in large radius limit. However, without knowledge of the actual mirror map, we do not know which part of the complex structure moduli space is mapped to the large radius limit. We will have to conjecture the large radius limit is the region of $\mathcal{C}_Y$ where the monomial-divisor map is well defined. It turns out that in this conjectured large radius limit, there are Q fixed points. Hence, we tune the Kähler parameters to one of those points, and apply the monomial-divisor map to find out Q-action on the mirror. In this section, we describe the tangent spaces of $\mathcal{K}_{\widehat{Y}}$ and $\mathcal{C}_Y$ and the monomial-divisor mirror map following [1]. The authors of [1] only considered Calabi-Yau manifolds that are hypersurfaces in toric varieties. We will need to extend the argument a little bit to suit complete intersection Calabi-Yau cases.

2.3.1 Divisors

We describe the tangent space of $\mathcal{K}_{\widehat{Y}}$. First, we resolve all singularities of $\widehat{V}$.[e] We add new edges to the fan $\widehat{\Sigma}$ and subdivide cones so that the resulting fan is simplicial. In this way, we get the blown up $\widetilde{V}$ of $\widehat{V}$. There is an induced blown up $\widetilde{Y}$ of $\widehat{Y}$, too. The new edges will be generated by lattice points of ∇^*. Since ∇^* is reflexive, it contains only one interior lattice point, 0: the other lattice points are on faces of ∇^*. Let's call Ξ the set of these points:

$$\Xi = \nabla^* \cap M - \{0\}. \tag{2.39}$$

[e] It is not necessary to resolve all singularities since some miss $\widehat{Y}$. We only need to resolve singularities of $\widehat{Y}$. However, it will turn out to be more useful in understanding the monomial-divisor map to resolve all.

It turns out that every lattice point in ∇^* is in one of Δ_l's. This is a necessary condition for the monomial-divisor mirror map to be well defined for complete intersection Calabi-Yau cases. We do not know if this condition is generally met for other cases. Ξ contains 140 points: $\omega_i + \omega_j - \omega_{2l-1} - \omega_{2l}$. Label them $\xi_{l,i,j}$ and denote by $\widetilde{D}_{l,i,j}$ the associated divisors. There is an isomorphism between $\mathbb{Z}^\Xi$, the set of integer-valued functions on Ξ, and $\mathrm{WDiv}_T(\widetilde{V})$, the set of T-stable Weil divisors in $\widetilde{V}$. Given an integer-valued function ϕ on Ξ, one can construct a Weil divisor, $\sum \phi(\xi_{l,i,j})\widetilde{D}_{l,i,j}$. The Chow group $A_6(\widetilde{V})$ is the quotient of $\mathrm{WDiv}_T(\widetilde{V})$ by linear equivalence. As M represents the set of meromorphic functions on $\mathbb{P}^7$, meromorphic functions on $\widetilde{V}$ will be represented by elements in N. For $n \in N$, we denote the corresponding function by $\widetilde{\chi}^n$. Two linearly equivalent Weil divisors, W, W' are related to each other by

$$W - W' = \sum \langle n, \xi_{l,i,j} \rangle \widetilde{D}_{l,i,j} \tag{2.40}$$

for some $n \in N$. Equivalently, consider embedding $\mathrm{ad}_\Xi : N \to \mathbb{Z}^\Xi$ by sending $n \in N$ to the function $\mathrm{ad}_\Xi(n)$ defined by $\mathrm{ad}_\Xi(n) : \xi_{l,i,j} \mapsto \langle n, \xi_{l,i,j} \rangle$. Then,

$$\begin{aligned} A_6(\widetilde{V}) &\simeq \mathbb{Z}^\Xi / N \\ \mathrm{H}^{1,1}(\widetilde{V}) &\simeq \left(\mathbb{Z}^\Xi / N\right) \otimes \mathbb{C}. \end{aligned} \tag{2.41}$$

The tangent space of $\widehat{\mathcal{K}}$ is not $\mathrm{H}^{1,1}(\widetilde{V})$ but $\mathrm{H}^{1,1}(\widetilde{Y})$. Let's consider $\mathrm{WDiv}_T(\widetilde{Y})$. Generally, a divisor in $\widetilde{V}$ induces a divisor in $\widetilde{Y}$. However, some divisors in $\mathrm{WDiv}_T(\widetilde{V})$ do not intersect $\widetilde{Y}$. We need to exclude those divisors. Detailed investigation of $\widehat{\Sigma}$ shows that the following 68 divisors do not intersect

$\widetilde{Y}$ for $l, m = 1, \ldots, 4$, $i = 1, \ldots, 8$ and $i \neq 2l-1, 2l$ and $m \neq l$:

$$\widetilde{D}_{l,2l-1,i} \quad \widetilde{D}_{l,2l,i} \quad \widetilde{D}_{l,2m-1,2m}. \tag{2.42}$$

Let Ξ^0 be the set of lattice points representing divisors that do intersect $\widetilde{Y}$. Ξ^0 contains 72 lattice points. As before, $\mathrm{WDiv}_T(\widetilde{Y}) \simeq \mathbb{Z}^{\Xi^0}$, and the linear equivalence is given by the embedding of N in $\mathbb{Z}^{\Xi^0}$. Therefore,[f]

$$\mathrm{H}^{1,1}(\widetilde{Y}) \simeq \left(\mathbb{Z}^{\Xi^0}/N\right) \otimes \mathbb{C}. \tag{2.43}$$

2.3.2 Monomials

The tangent space of $\mathcal{C}_Y$ is $\mathrm{H}^{2,1}(Y)$. The simplest way to deform the complex structure of Y is to perturb the defining equations, $G_1, \ldots, G_4$. Generally speaking, there might be deformations that cannot be realized in this way [32]. In our case, the dimensional analysis tells us there are no such deformations. As we explained in the previous section, the monomials in equation G_l are represented by the lattice points in Δ_l. Let $\mathcal{A}$ be the space of 4 quadrics in $\mathbb{P}^7$. Then,

$$\mathcal{A} = \bigotimes_{l=1}^{4} \mathbb{C}^{\Delta_l \cap M} \tag{2.44}$$

Here, each factor $\mathbb{C}^{\Delta_l \cap M}$ represents the coefficients of the monomials in G_l. Not every point of $\mathcal{A}$ will give distinct choice of the complex structure. Linear

[f]Sometimes, the toric divisors do not generate the entire Chow group. In our case, we know that $h^{1,1}(\widetilde{Y}) = 65$ by mirror symmetry. Since $\dim\left[\left(\mathbb{Z}^{\Xi^0}/N\right) \otimes \mathbb{C}\right] = 65$, we conclude that the above expression is indeed correct

transformations of G_l's will define the same manifold Y. Also, linear transformations of X_i's will give manifolds isomorphic to the original manifold Y. From the gauge sigma model analysis, we know that infinitesimal changes in the superpotential W that are generated by its derivatives will give an isomorphic theory. Since $W = P_1 G_1 + \ldots + P_4 G_4$, such changes are

$$\delta W = \alpha_{m,l} P_l \frac{\partial W}{\partial P_m} + \beta_{j,i} X_i \frac{\partial W}{\partial X_j}. \tag{2.45}$$

They are generated by the infinitesimal linear transformations of P_l and X_i. Some elements of $\mathrm{GL}(4, \mathbb{C}) \otimes \mathrm{GL}(8, \mathbb{C})$ act trivially on $\mathcal{A}$. They are

$$\begin{aligned} G_l &\mapsto \lambda^{-2} G_l \\ X_i &\mapsto \lambda X_i \end{aligned} \tag{2.46}$$

for some $\lambda \in \mathbb{C}^*$.[g] Hence, the actual group acting on $\mathcal{A}$ is

$$\mathcal{G} = \left(\mathrm{GL}(4, \mathbb{C}) \otimes \mathrm{GL}(8, \mathbb{C})\right) / \mathbb{C}^*. \tag{2.47}$$

Therefore, we find that

$$\mathcal{C}_Y \simeq \mathcal{A} / \left(\mathcal{G} \otimes \Gamma\right) \tag{2.48}$$

where Γ is the discrete group that consists of diffeomorphisms of Y not connected to the identity, hence not captured in the above discussion.

We would like to describe the tangent space of $\mathcal{C}_Y$ using toric geometry. Unfortunately, not every element of $\mathcal{G}$ is compatible with the toric description.

[g] As we will see later, at special points in $\mathcal{A}$, $\mathrm{GL}(4, \mathbb{C}) \otimes \mathrm{GL}(8, \mathbb{C})$ may have additional elements that act trivially. But, there are only a finite number of them and they do not change our arguments below.

First, some elements of $\mathrm{GL}(8,\mathbb{C})$ are not compatible with the T-action, the torus action on the toric variety. As we have seen in the previous section, T-equivariant Cartier divisors are monomials. Generally, linear transformations of X_i's map monomials to polynomials. Such linear transformations are not compatible with the T-action. Second, G_l's are represented by polyhedrons Δ_i and it is not clear how to represent linear transformations of G_l's in terms of polyhedrons. However, there are two kinds elements of $\mathcal{G}$ that are compatible with T. They are scalings

$$\begin{aligned} G_l &\mapsto \rho_l G_l \quad \text{(no sum)} \\ X_i &\mapsto \lambda_i X_i \quad \text{(no sum)} \end{aligned} \tag{2.49}$$

and permutations:

$$\begin{aligned} G_l &\mapsto G_{\pi(l)} \\ X_i &\mapsto X_{\pi'(i)} \end{aligned} \tag{2.50}$$

where π, π' are elements of S_4 and S_8, the permutation group of 4 and 8 objects, respectively. In finding the toric description of the tangent space of $\mathcal{C}_Y$, only scalings will play an important role. Taking account of the elements acting trivially on $\mathcal{A}$, we find that scalings comprise a subgroup $\mathcal{G}_T$ isomorphic to

$$T \otimes (\mathbb{C}^*)^4. \tag{2.51}$$

Since only part of $\mathcal{G}$ is realized in the toric description, we will use the rest of $\mathcal{G}$ to fix some coefficients of the monomials in G_l. This is a tricky procedure. Generally, the choice of monomials whose coefficients will be fixed depends on where you are on the moduli space. Since we are interested in finding the monomial-divisor mirror map, we would like to be in the large radius limit. However, without knowledge of the actual mirror map, we do not know which part of $\mathcal{A}$ is mapped to the large radius limit. We need to guess. Mirror symmetry tells us that the tangent space of $\mathcal{C}_Y$ in the large radius limit is isomorphic to

$$\left(\mathbb{Z}^{\Xi^0}/N\right) \otimes \mathbb{C}. \tag{2.52}$$

Notice the dual role the lattice points in Δ_l play here. The lattice point $\xi_{l,i,j} = \omega_i + \omega_j - \omega_{2l-l} - \omega_{2l}$ represents the monomial $X_i X_j$ in G_l. It also represents toric divisor $\widetilde{D}_{l,i,j}$ of $\widetilde{V}$ on the mirror side. There is a subtlety. While 0 in Δ_l represents monomial $X_{2l-1}X_{2l}$ of G_l, there is no divisor corresponding to this point. Also, unlike other points, 0 represents 4 monomials, one from each G_l since it is the only common point of $\Delta_1, \ldots, \Delta_4$. Using the fact that $\Xi = \bigcup_{l=1}^{4} (\Delta_l \cap M) - \{0\}$, we have

$$\mathcal{A} \simeq \mathbb{C}^{\Xi} \otimes \mathbb{C}^4 \tag{2.53}$$

where the extra factor $\mathbb{C}^4$ represents the coefficient of the 4 monomials that the point 0 represents. This $\mathbb{C}^4$ will be cancelled by $(\mathbb{C}^*)^4$ in (2.51). Remember that the factor $(\mathbb{C}^*)^4$ represents scaling:

$$G_l \mapsto \rho_l G_l \quad \text{(no sum)}. \tag{2.54}$$

We can use this scaling to set the coefficient of $X_{2l-1}X_{2l}$ in G_l to 1 as long as the original coefficients are non-zero. There are other coefficients to fix. Some of divisors in $\widetilde{V}$ do not intersect $\widetilde{Y}$. Only divisors corresponding to the points in Ξ^0 intersect $\widetilde{Y}$. Therefore, it is tempting to fix the coefficients of the monomials corresponding to the points in Ξ which are not in Ξ^0. There are 68 such points. Since the dimension of $\mathcal{G}/\mathcal{G}_T$ is 68, it is plausible that we could fix these 68 coefficients. This time, we want to set these coefficients to zero so that $\mathcal{G}_T$ does not change them. Hence, we conjecture that in the large radius limit of $\mathcal{A}$, there is an element of $\mathcal{G}$ that we can use to set

- the coefficient of $X_{2l-1}X_{2l}$ of G_l to 1
- the coefficient of monomials:

$$X_{2l-1}X_i, \quad X_{2l}X_i, \quad X_{2m-1}X_{2m} \qquad i \neq 2l-1, 2l \text{ and } m \neq l \tag{2.55}$$

  of G_l to zero .

Once we fixed the coefficients, the conjectured large radius limit of $\mathcal{C}_Y$ looks like

$$\mathbb{C}^{\Xi^0}/T \tag{2.56}$$

and the tangent space $\mathrm{H}^{2,1}(Y)$ is

$$\mathbb{C}^{\Xi^0}/\left(N \otimes \mathbb{C}\right). \tag{2.57}$$

Here, we have used the fact that the tangent space of T is $N \otimes \mathbb{C}$.[h] Now, the monomial-divisor mirror map is evident since

$$\left(\mathbb{Z}^{\Xi^0}/N\right) \otimes \mathbb{C} \simeq \mathbb{C}^{\Xi^0}/\left(N \otimes \mathbb{C}\right) \tag{2.58}$$

[h]This can be a little confusing since $T = N \otimes_{\mathbb{Z}} \mathbb{C}^*$. The group multiplication is the addition in N and the multiplication in $\mathbb{C}^*$. Now, we can represent $\mathbb{C}^*$ by $\exp \mathbb{C}$. Then, $T = \exp N \otimes \mathbb{C}$.

2.4 Mirror of the Beauville manifold

We will construct the mirror of the Beauville manifold by taking a quotient $\widetilde{X} = \widetilde{Y}/Q$. To do so, first, we need to find how Q acts on $\widetilde{Y}$.

2.4.1 Mirror Q-action

We tune our parameters to one of the Q-fixed points of $\mathcal{C}_Y$. It amounts to choosing 4 quadrics, one from each one-dimensional irreducible representation of Q. Y defined with this choice of the quadrics, $G_1, \ldots, G_4$, is invariant under Q because the induced Q-action on quadrics will transform G_l's to some linear combinations of them. We can embed Q into $\mathrm{GL}(4,\mathbb{C})$ such that this embedded Q takes the transformed G_l's back to the original. In other words, at these points, there are additional elements of $\mathrm{GL}(4,\mathbb{C}) \otimes \mathrm{GL}(8,\mathbb{C})$ that act trivially on $\mathcal{A}$ and they comprise a subgroup isomorphic to Q. Actual embedding of Q is determined by the choice of the homogeneous coordinates and the quadrics that give the same Y. We want this embedding to be compatible with the toric description. Since elements of $\mathcal{G}$ compatible with the toric description are scalings and permutations, we need to find the choice of the homogeneous coordinates and the quadrics that embeds Q into the scalings and permutations.

One might wonder why it matters because every choice of the homogeneous coordinates and the quadrics is related to each other by linear transformations and any choice is as good as any other. However, this is not true. First, linear transformations of the homogeneous coordinates and the quadrics will map scalings and permutations to other linear transformations that are not compatible with the toric description. Second, to get the toric description of the tangent space of $\mathcal{C}_Y$, we have fixed some coefficients of the quadrics. If we linear-transform the homogeneous coordinates or the quadrics, it will change the fixed coefficients and ruin the toric description and hence, also the monomial-divisor mirror map.

It turns out that the right choice will embed Q into permutations. We choose the homogeneous coordinates such that Q-action is given by:

$$g_1 \cdot X_{g_2} = X_{g_1 g_2} \tag{2.59}$$

where $g_1, g_2 \in Q$ and we have relabeled the homogeneous coordinates X_g. To explain the choice of the quadrics, we note that $V_4 = V_1 \oplus V_I \oplus V_J \oplus V_K$ is the regular representation of the abelianization $Q/[Q,Q] = \mathbb{Z}_2 \times \mathbb{Z}_2$. Elements of $Q/[Q,Q]$ are the cosets of the normal subgroup, $[Q,Q] = \{1,-1\}$. We denote by $[g]$ the coset $g \cdot [Q,Q]$ for each $g \in Q$. Then $[g] = [-g]$ and the multiplication law is given by $[g_1][g_2] = [g_1 g_2]$. We label the quadrics, $G_{[g]}$ and choose them such that Q acts on them as

$$g_1 \cdot G_{[g_2]} = G_{[g_1 g_2]} \tag{2.60}$$

where $g_1, g_2 \in Q$. With this choice of the homogeneous coordinates and the quadrics, it is clear that Q acts on them as permutations. Now, let's write down the most general $G_{[g]}$'s and see if there is a fixed point in the conjectured large radius limit. The most general $G_{[g]}$'s are

$$
\begin{aligned}
G_{[1]} &= t_1 X_1 X_{-1} + t_2 X_I X_{-I} + t_3 X_J X_{-J} + t_4 X_K X_{-K} \\
&+ t_5(X_1^2 + X_{-1}^2) + t_6(X_I^2 + X_{-I}^2) + t_7(X_J^2 + X_{-J}^2) + t_8(X_K^2 + X_{-K}^2) \\
&+ t_9(X_1 X_I + X_{-1} X_{-I}) + t_{10}(X_1 X_J + X_{-1} X_{-J}) + t_{11}(X_1 X_K + X_{-1} X_{-K}) \\
&+ t_{12}(X_1 X_{-I} + X_{-1} X_I) + t_{13}(X_1 X_{-J} + X_{-1} X_J) + t_{14}(X_1 X_{-K} + X_{-1} X_K) \\
&+ t_{15}(X_I X_J + X_{-I} X_{-J}) + t_{16}(X_J X_K + X_{-J} X_{-K}) + t_{17}(X_K X_I + X_{-K} X_{-I}) \\
&+ t_{18}(X_I X_{-J} + X_{-I} X_J) + t_{19}(X_J X_{-K} + X_{-J} X_K) + t_{20}(X_K X_{-I} + X_{-K} X_I) \\
G_{[I]} &= t_1 X_I X_{-I} + t_2 X_1 X_{-1} + t_3 X_K X_{-K} + t_4 X_J X_{-J} \\
&+ t_5(X_I^2 + X_{-I}^2) + t_6(X_1^2 + X_{-1}^2) + t_7(X_K^2 + X_{-K}^2) + t_8(X_J^2 + X_{-J}^2) \\
&+ t_9(X_1 X_{-I} + X_{-1} X_I) + t_{10}(X_I X_K + X_{-I} X_{-K}) + t_{11}(X_I X_{-J} + X_{-I} X_J) \\
&+ t_{12}(X_1 X_I + X_{-1} X_{-I}) + t_{13}(X_K X_{-I} + X_{-K} X_I) + t_{14}(X_I X_J + X_{-I} X_{-J}) \\
&+ t_{15}(X_1 X_{-K} + X_{-1} X_K) + t_{16}(X_J X_{-K} + X_{-J} X_K) + t_{17}(X_1 X_J + X_{-1} X_{-J}) \\
&+ t_{18}(X_1 X_K + X_{-1} X_{-K}) + t_{19}(X_J X_K + X_{-J} X_{-K}) + t_{20}(X_1 X_{-J} + X_{-1} X_J) \\
G_{[J]} &= t_1 X_J X_{-J} + t_2 X_K X_{-K} + t_3 X_1 X_{-1} + t_4 X_I X_{-I} \\
&+ t_5(X_J^2 + X_{-J}^2) + t_6(X_K^2 + X_{-K}^2) + t_7(X_1^2 + X_{-1}^2) + t_8(X_I^2 + X_{-I}^2) \\
&+ t_9(X_J X_{-K} + X_{-J} X_K) + t_{10}(X_1 X_{-J} + X_{-1} X_J) + t_{11}(X_I X_J + X_{-I} X_{-J}) \\
&+ t_{12}(X_J X_K + X_{-J} X_{-K}) + t_{13}(X_1 X_J + X_{-1} X_{-J}) + t_{14}(X_I X_{-J} + X_{-I} X_J) \\
&+ t_{15}(X_1 X_K + X_{-1} X_{-K}) + t_{16}(X_1 X_{-I} + X_{-1} X_I) + t_{17}(X_K X_{-I} + X_{-K} X_I) \\
&+ t_{18}(X_1 X_{-K} + X_{-1} X_K) + t_{19}(X_1 X_I + X_{-1} X_{-I}) + t_{20}(X_K X_I + X_{-K} X_{-I})
\end{aligned}
$$

$$
\begin{aligned}
G_{[K]} &= t_1 X_K X_{-K} + t_2 X_J X_{-J} + t_3 X_I X_{-I} + t_4 X_1 X_{-1} \\
&+ t_5(X_K^2 + X_{-K}^2) + t_6(X_J^2 + X_{-J}^2) + t_7(X_I^2 + X_{-I}^2) + t_8(X_1^2 + X_{-1}^2) \\
&+ t_9(X_J X_K + X_{-J} X_{-K}) + t_{10}(X_K X_{-I} + X_{-K} X_I) + t_{11}(X_1 X_{-K} + X_{-1} X_K) \\
&+ t_{12}(X_J X_{-K} + X_{-J} X_K) + t_{13}(X_K X_I + X_{-K} X_{-I}) + t_{14}(X_1 X_K + X_{-1} X_{-K}) \\
&+ t_{15}(X_I X_{-J} + X_{-I} X_J) + t_{16}(X_1 X_I + X_{-1} X_{-I}) + t_{17}(X_1 X_{-J} + X_{-1} X_J) \\
&+ t_{18}(X_I X_J + X_{-I} X_{-J}) + t_{19}(X_1 X_{-I} + X_{-1} X_I) + t_{20}(X_1 X_J + X_{-1} X_{-J})
\end{aligned} \tag{2.61}
$$

We choose the following nef partition:

$$
\Delta_{[g]} = \{e_g, e_{-g}\} \tag{2.62}
$$

where e_g is the vertex of Δ^* that represents the divisor $D_g = \{X_g = 0\}$. Then, there are Q-fixed points in the large radius limit. They are points with $t_1 = 1, t_2 = t_3 = t_4 = t_5 = t_9 = t_{10} = t_{11} = t_{12} = t_{13} = t_{14} = 0$.

Now, we tune our parameters to one of these points and apply the monomial-divisor mirror map to find the mirror Q-action on $\widetilde{Y}$. Actually, it turns out that $\widetilde{V}$ has a Q-action too and it is easy to identify. To do so, we introduce the homogeneous coordinates on $\widetilde{V}$

$$
\widetilde{X}_{[g_1],g_2,g_3} \tag{2.63}
$$

where $[g_1] \in Q/[Q,Q]$ and $g_2, g_3 \in Q$. Since we have relabeled the homogeneous coordinates and the quadrics, we relabel everything on the mirror side accordingly. As usual, $\widetilde{X}_{[g_1],g_2,g_3}$ is determined up to a multiplicative constant by $\widetilde{D}_{[g_1],g_2,g_3} = \left\{\widetilde{X}_{[g_1],g_2,g_3} = 0\right\}$. $\widetilde{A}$ also has the holomorphic quotient description

$$\widetilde{V} \simeq \frac{\mathbb{C}^{140} - F_{\widetilde{\Sigma}}}{\widetilde{G}}. \tag{2.64}$$

The group $\widetilde{G}$ is defined similar to G:

$$\widetilde{X}_{[g_1],g_2,g_3} \mapsto \Lambda_{[g_1],g_2,g_3}\widetilde{X}_{[g_1],g_2,g_3} \quad \text{(no sum)}. \tag{2.65}$$

Here, $\Lambda_{[g_1],g_2,g_3}$'s are non-zero complex numbers with

$$1 = \prod_{\substack{[g_1]\in Q/[Q,Q] \\ g_2,g_3\in Q}} \Lambda_{[g_1],g_2,g_3}^{\langle n,\xi_{[g_1],g_2,g_3}\rangle} \qquad \forall n \in N. \tag{2.66}$$

It is easy to show $\widetilde{G} \simeq (\mathbb{C}^*)^{133}$.

Since we are at one of the Q-fixed points of $\mathcal{A}$, we have Q-action on its tangent space $T\mathcal{A}$. Note

$$T\mathcal{A} = \mathbb{C}^{\Xi} \oplus \mathbb{C}^4. \tag{2.67}$$

Here, the term $\mathbb{C}^4$ corresponds to the coefficients of the 4 monomials that the point 0 represents and is invariant under the Q-action. Q permutes these 4 monomials. Each point $\xi \in \Xi$ represents both a monomial in one of the quadrics and a T-stable Weil divisor of $\widetilde{V}$. Since we know how Q permutes monomials, we can deduce the Q-action on $\mathrm{WDiv}_T(\widetilde{V})$:

$$g \cdot \widetilde{D}_{[g_1],g_2,g_3} = \widetilde{D}_{[gg_1],gg_2,gg_3}. \tag{2.68}$$

In terms of the homogeneous coordinates

$$g \cdot \widetilde{X}_{[g_1],g_2,g_3} = \widetilde{X}_{[gg_1],gg_2,gg_3}. \tag{2.69}$$

Since this Q-action maps polyhedron ∇^* to itself, it is consistent with the construction of the toric variety $\widetilde{V}$.

2.4.2 Mirror of the Beauville manifold

To define $\widetilde{X} = \widetilde{Y}/Q$, we need to show $\widetilde{Y}$ is invariant under Q and Q acts freely on $\widetilde{Y}$. The easiest way of showing this is to go back to $\widehat{Y}$ and work there. Q permutes the vertices of ∇^*. Therefore, $\widehat{V}$ has the Q-action too:

$$g \cdot \widehat{X}_{[g_1],g_2} = \widehat{X}_{[gg_1],gg_2} \tag{2.70}$$

One can easily check that this Q-action permutes $\widehat{G}_{[g]}$'s:

$$\widehat{G}_{[g_1]}(g \cdot \widehat{X}) = \widehat{G}_{[gg_1]}(\widehat{X}). \tag{2.71}$$

This implies $\widehat{Y}$ is invariant under Q. Now, we would like to show that Q acts freely on $\widehat{Y}$. Suppose there is a point $p \in \widehat{Y}$ that is fixed by some non-identity elements of Q. Then, p is also fixed by $-1 \in Q$. Note that the values of the homogeneous coordinates are fixed by -1 up to a gauge transformation. Hence, at p, we have

$$\widehat{X}_{[g_1],g_2} = \Lambda_{[g_1],g_2} \widehat{X}_{[g_1],-g_2} \quad \text{(no sum)} \tag{2.72}$$

for some $\Lambda_{[g_1],g_2} \in \mathbb{C}^*$ satisfying (2.26). Plugging the above equation in (2.26), we get

$$\prod_{[g'] \in Q/[Q,Q]} \widehat{X}^2_{[g'],g} = \prod_{g' \in Q/[Q,Q]} \widehat{X}^2_{[g'],-g} \tag{2.73}$$

for all $g \in Q$. With (2.31), this implies

$$(\psi^8 - 1) \prod_{\substack{[g_1] \in Q/[Q,Q] \\ g_2 \in Q}} \widehat{X}^2_{[g_1]g_2} = 0. \tag{2.74}$$

By closely investigating $\widehat{\Sigma}$, one can show[i]

- $X_{[g],g}$ and $X_{[g],-g}$ do not intersect $\widehat{G}_{[g]}$ (and hence $\widehat{Y}$).
- For $[g_1] \neq [g_2]$, $\left\{\widehat{X}_{[g_1],g_2} = \widehat{X}_{[g_1],-g_2} = 0\right\}$ does not intersect $\widehat{G}_{[g_2]}$ (and hence $\widehat{Y}$).

Together with (2.72), it is easy to see no $\widehat{X}_{[g_1],g_2}$ vanishes at p. Therefore, there is no solution to (2.74) on $\widehat{Y}$ unless $\psi^8 = 1$. As we will see later, $\psi^8 = 1$ is the conifold point. For generic value of ψ, there is no such p, and Q acts freely on $\widehat{Y}$. Since $\widetilde{Y}$ is the blow-up of $\widehat{Y}$, we conclude that $\widetilde{Y}$ is invariant under Q and the Q-action on it is free too. We take the quotient of $\widetilde{Y}$ by Q to define the mirror $\widetilde{X}$ of the Beauville manifold.

Let's take a close look at the moduli spaces of X and $\widetilde{X}$. As in the case of Y, the complex structure moduli space, $\mathcal{C}_X$, of X can be described as a quotient of the space, $\mathcal{A}_X$ of 4 quadrics defining X by the group $\mathcal{G}_X$ generating the manifolds that are isomorphic to X. The most general 4 quadrics defining X have been written in (2.61). From that,

$$\mathcal{A}_X \simeq \mathbb{C}^{20} \tag{2.75}$$

Now, we claim that

$$\mathcal{G}_X \simeq (\mathbb{C}^*)^7 \otimes \mathrm{GL}(2,\mathbb{C}) \otimes \Gamma_X \tag{2.76}$$

[i] One way to show this is the following. In [10], it is shown that $\nabla = \nabla_1 + \ldots + \nabla_4$. With this, one can easily identify the full dimensional cones in $\widehat{\Sigma}$. The above follows from the the fact that divisors have common points only if they all belong to a same full dimensional cone in $\widehat{\Sigma}$ [27].

where Γ_X is a discrete group whose elements are diffeomorphisms of X not connected to the identity. To see this, notice [j]

$$\begin{aligned} V_8 &= V_1 \oplus V_I \oplus V_J \oplus V_K \oplus 2V_2 \\ V_4 &= V_1 \oplus V_I \oplus V_J \oplus V_K. \end{aligned} \tag{2.77}$$

From this decomposition, it is clear that the factor $(\mathbb{C}^*)^4 \otimes \mathrm{GL}(2, \mathbb{C})$ comes from linear transformations of X_g's and the factor $(\mathbb{C}^*)^4$ from linear transformations of $G_{[g]}$'s. As before, they have the subgroup $\mathbb{C}^*$ that acts trivially on $\mathcal{A}_X$ proving the claim. Out of $\mathcal{G}_X$, only $\mathbb{C}^*$ has the toric description. It corresponds to the overall scaling of the homogeneous coordinates. With this $\mathbb{C}^*$, we set t_1 in (2.61) to 1. The rest of $\mathcal{G}_X$ will be used to set $t_2, t_3, t_4, t_5, t_9, t_{10}, t_{11}, t_{12}, t_{13}, t_{14}$ to 0 (in the large radius limit). Therefore, the tangent space of $\mathcal{C}_X$ in the large radius limit is

$$T\mathcal{C}_X = \mathbb{C}^9 \tag{2.78}$$

where $\mathbb{C}^9$ is parameterized by 9 t_i's that are not fixed.

The tangent space of Kähler moduli space $\mathcal{K}_{\widetilde{X}}$ of the mirror $\widetilde{X}$ is

$$T\mathcal{K}_{\widetilde{X}} = \Big(\mathrm{WDiv}_T(\widetilde{X})/\text{linear equivalence}\Big) \otimes \mathbb{C}. \tag{2.79}$$

For a given T-stable Weil divisor of $\widetilde{X}$, one finds a Q-invariant toric divisor in $\widetilde{Y}$ via the pull-back of the projection: $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$. Q-invariant Weil divisors of $\widetilde{Y}$ are generated by the following elements:

$$\sum_{g' \in Q} \widetilde{D}_{[g'], g'g_1, g'g_2} \tag{2.80}$$

[j]In the previous subsection, we could have used the homogeneous coordinates and the quadrics that make this decomposition manifest. This was the choice Beauville originally used in [7]. However, this choice would have given no fixed points in the large radius limit.

where $g_1, g_2 = \pm I, \pm J, \pm K$ and $g_1 \neq -g_2$. The number of such elements is 9. Hence,

$$\mathrm{WDiv}_T(\widetilde{X}) \simeq \mathbb{Z}^9. \tag{2.81}$$

The linear equivalence on $\mathrm{WDiv}_T(\widetilde{X})$ is trivial. Linearly equivalent divisors will induce the linearly equivalent divisors by the pull-back, $\widetilde{\pi}^*$. Note

$$\sum_{g' \in Q} \xi_{[g'],g'g_1,g'g_2} = 0 \tag{2.82}$$

for any $g_1, g_2 \in Q$. This implies that no non-zero Q-invariant toric divisor in $\widetilde{Y}$ is linearly trivial and hence,

$$T\mathcal{K}_{\widetilde{X}} \simeq \mathbb{Z}^9 \otimes \mathbb{C} \simeq \mathbb{C}^9. \tag{2.83}$$

We conclude this section by mentioning that Q-action on $\widetilde{V}$ will induce the original Q-action on $\mathbb{P}^7$ via the monomial-divisor mirror map between the complex structure moduli space of $\widetilde{Y}$ and the Kähler moduli space of Y. The lattice points of $\nabla_{[g]}$ have the dual role as before: representing monomials in $\widetilde{G}_{[g]}$ and divisors in $\mathbb{P}^7$. Each $\nabla_{[g]}$ has three such points: $0, e_g, e_{-g}$. The only common point 0 represents 4 monomials, one from each $\widetilde{G}_{[g]}$. Q-action on the homogeneous coordinates $\widetilde{X}_{[g_1],g_2,g_3}$ of $\widetilde{V}$ will permute these 4 monomials. Lattice points representing other monomials in $\widetilde{G}_{[g]}$ will be mapped to each other by Q-action on $\widetilde{X}_{[g_1],g_2,g_3}$ giving the original Q-action on Y.

2.5 Application

Q has a 2-dimensional irreducible representation, V_2, out of which one can build a flat rank-2 vector bundle on X. In [12], it was conjectured that there exists a threshold bound state of 2 D6-branes corresponding to this vector bundle and this state becomes massless around the "conifold" point in the Kähler moduli space of X. In addition to this state, there are 4 more states that become massless. They are the line bundles associated to 1-dimensional irreducible representations, V_1, V_I, V_J, and V_K. Unlike these line bundles, the existence of the flat rank-2 vector bundle as a stable single-particle state was not guaranteed by the BPS condition (it is degenerate with a pair of D6-branes), nor by K-theory (it does not carry any K-theory charge by which it might be distinguished from a pair of D6-branes). It is hard to check the existence directly on X since we are far from the large radius limit. Mirror symmetry gives a way of confirming it because the classical consideration on $\widetilde{X}$ is exact.

First, let's consider the complex structure moduli space of $\widetilde{X}$ and find the conifold point. Since it is independent of where we are in the Kähler moduli space, we can work on $\widehat{X} = \widehat{Y}/Q$, instead. It turns out that ψ^8 is the invariant parameter. To see this consider the following linear transformation:

$$\widehat{X}_{[g],1} \mapsto \zeta \widehat{X}_{[g],1} \tag{2.84}$$

with $\zeta^8 = 1$ and other homogeneous coordinates fixed. This transformation commutes with the Q-action modulo gauge transformation (2.65). This will change ψ to $\zeta\psi$. It is not difficult to show that ψ^8 is invariant under linear transformations that keep $\widehat{G}_{[g]}$'s in the form (2.31). This is consistent with the discussion we had earlier. $\widehat{Y}$'s invariant parameter is also ψ^8. Therefore, the moduli space of $\widehat{X}$ is the same as that of $\widehat{Y}$.

It is obvious that $\widehat{X}$ becomes singular at $\psi = 0, \infty$. They are the hybrid point and the large radius limit of X. There is another value of ψ^8 where $\widehat{X}$ becomes singular. At singular points on the complex moduli space of $\widehat{X}$, there exist non-trivial solutions $P_{[g]}$ to the following equation:

$$\sum_{[g]\in Q/[Q,Q]} P_{[g]} \mathrm{d}\widehat{G}_{[g]} = 0. \tag{2.85}$$

By non-triviality, we mean that not every $P_{[g]}$ is zero. With assumption $\psi \neq 0, \infty$, one can show that the previous equation implies

$$\prod_{[g']\in Q/[Q,Q]} \widehat{X}^2_{[g'],g} = \prod_{[g']\in Q/[Q,Q]} \widehat{X}^2_{[-g'],-g}. \tag{2.86}$$

Recall that this was the Q-fixed point condition in (2.72). Therefore, the transversality fails at Q-fixed points and it happens only when $\psi^8 = 1$. The detailed calculation shows that there is only one such point in $\widehat{X}$ and it is given by $\widehat{X}_{[g],g'} = 1$ when $\psi = 1$. This is the conifold point of $\widehat{X}$. To see how many states become massless, we expand around this point. Since non of the homogeneous coordinates is zero, we can make the following gauge choice:

$$\widehat{X}_{[1],1} = \widehat{X}_{[g],g'} = 1 \tag{2.87}$$

where $[g] \neq [1]$. Basically, we gauged away all homogeneous coordinates but $\widehat{X}_{[1],g}, g \neq 1$. Around the conifold point, we set

$$\begin{gathered} \psi = 1 + \frac{\epsilon}{8} \\ \widehat{X}_{[1],-1} = 1 + y_1 \\ \widehat{X}_{[1],I} = \sqrt{\psi}(1 + \frac{y_2}{2}) \ , \ \ldots \ , \ \widehat{X}_{[1],-K} = \sqrt{\psi}(1 + \frac{y_7}{2}). \end{gathered} \tag{2.88}$$

Assuming $\epsilon \ll 1$ and $y_i \sim \mathcal{O}(\sqrt{\epsilon})$, we get the following from $\widehat{G_1}, \ldots, \widehat{G_4}$ (up to the first order in ϵ).

$$
\begin{aligned}
\epsilon &= y_1^2 + y_3^2 + y_5^2 + y_7^2 \\
y_2 &= -y_3 - y_3^2 \\
y_4 &= -y_5 - y_5^2 \\
y_6 &= -y_7 - y_7^2
\end{aligned}
\tag{2.89}
$$

This is the cotangent space of S^3. If ϵ is real positive, then S^3 is parameterized by the real part of y_1, y_3 and y_5. Now, let's consider how Q acts on it. Q-action must be followed by appropriate gauge transformations to maintain the gauge choice (2.87). Since I and J generate the entire Q, it is enough to describe their action:

$$
\begin{aligned}
&I : y_1 \mapsto -y_3,\ y_3 \mapsto y_1,\ y_5 \mapsto -y_7,\ y_7 \mapsto y_5 \\
&J : y_1 \mapsto -y_5,\ y_3 \mapsto y_7\ , y_5 \mapsto y_1,\ y_7 \mapsto -y_3
\end{aligned}
\tag{2.90}
$$

This representation is isomorphic to $V_2 \oplus V_2$ and is real. Q maps S^3 into itself. Also, Q acts freely on S^3, since there is no Q-fixed point. Hence, the actual 3 cycle that shrinks to size zero is S^3/Q. Supersymmetric D3-branes wrapped on this 3 cycle have flat vector bundles. We can put 5 different vector bundles on S^3/Q; 4 from V_1, V_I, V_J and V_K and one from V_2. Therefore, there are 5 distinct states that become massless at the conifold point confirming the conjecture. Furthermore, from the mirror map, we know that the mirror of the flat rank-2 vector bundle on S^3/Q has D6-brane charge 2 and indeed it is the threshold bound state of 2 D6-branes.

Since we know which D-branes become massless at the conifold point, let's consider how the quantum symmetry group acts on them. Under tensor product, the flat line bundles form a group that is isomorphic to the quantum symmetry group. It is conjectured in [11], on the level of K-theory, that the quantum symmetry group acts on the D-brane charges by tensoring the flat line bundles. Note that this conjecture applies to both A-branes and B-branes. Therefore, it will be a good test to see if the conjectured action is compatible with mirror symmetry. On both X and $\widehat{X}$, we have 4 flat line bundles built from the representations, V_1, V_I, V_J and V_K. From the representation ring (2.4) we considered earlier, it is clear that the 4 states built from one-dimensional irreducible representations form the regular representation of the quantum symmetry group and the state with D6-brane charge 2 is invariant. This action is compatible with mirror symmetry.

Unfortunately, the only known example of Calabi-Yau manifolds with non-abelian fundamental groups is the Beauville manifold. With more examples, we can check more throughly the conjectures we considered here. Hence, finding more examples of such Calabi-Yau manifolds is desirable.

Chapter 3

D-Brane Monodromies and Derived Categories

In this chapter, we consider a class of D-branes on a Calabi-Yau manifold, X, which are in 1-1 correspondence with objects in $D(X)$, the derived category of coherent sheaves on X. We study the action of the monodromies in Kähler moduli space on these D-branes. We refine and extend a conjecture of Kontsevich about the form of one of the generators of these monodromies (the monodromy about the "conifold" locus) and show that one can do quite explicit calculations of the monodromy action in many examples.

3.1 The Derived Category

Here, we review derived categories. The main purpose of this section is to set up our notation. By no means is it intended to give a thorough review on the subject. For more rigorous treatment, see [28].

3.1.1 Construction of the Derived Category

We start with an Abelian category, $\mathbb{A}$. Examples of abelian categories – the ones which will be of most use to us – are the category of abelian groups, the category of finite dimensional vector spaces, and the category of coherent sheaves on a manifold X.

For a given Abelian category $\mathbb{A}$, one can construct several categories induced from it. The first category we will consider is $\mathrm{Kom}(\mathbb{A})$, *the category of complexes over* $\mathbb{A}$. Objects of $\mathrm{Kom}(\mathbb{A})$ are complexes in $\mathbb{A}$ and morphisms are chain maps. A *complex* $\mathcal{E}^\bullet$ in $\mathbb{A}$ is a sequence of objects and morphisms in $\mathbb{A}$

$$\mathcal{E}^\bullet : \cdots \xrightarrow{c_{n-1}} \mathcal{E}^n \xrightarrow{c_n} \mathcal{E}^{n+1} \xrightarrow{c_{n+1}} \cdots \tag{3.1}$$

with the property $c_n \circ c_{n-1} = 0$ for all n. In this paper, we will be exclusively interested in *bounded* complexes, where the $\mathcal{E}^n$ vanish except for a finite number of values of n. The corresponding category is usually denoted by $\mathrm{Kom}^b(\mathbb{A})$, but we will, for ease of notation, drop the "b" superscript; all our complexes will be bounded.

A *chain map* f from complex $\mathcal{E}^\bullet$:

$$\cdots \xrightarrow{c_{n-1}} \mathcal{E}^n \xrightarrow{c_n} \mathcal{E}^{n+1} \xrightarrow{c_{n+1}} \cdots$$

to complex $\mathcal{F}^\bullet$:

$$\cdots \xrightarrow{d_{n-1}} \mathcal{F}^n \xrightarrow{d_n} \mathcal{F}^{n+1} \xrightarrow{d_{n+1}} \cdots$$

is a family of morphisms $f^n \in \mathrm{Hom}_{\mathbb{A}}(\mathcal{E}^n, \mathcal{F}^n)$ satisfying $f^{n+1} \circ c^n = d^n \circ f^n$. This definition of chain map is nicely summarized in the following commutative diagram:

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{c_{n-1}} & \mathcal{E}^n & \xrightarrow{c_n} & \mathcal{E}^{n+1} & \xrightarrow{c_{n+1}} & \cdots \\
 & & {\scriptstyle f^n}\big\downarrow & & {\scriptstyle f^{n+1}}\big\downarrow & & \\
\cdots & \xrightarrow{d_{n-1}} & \mathcal{F}^n & \xrightarrow{d_n} & \mathcal{F}^{n+1} & \xrightarrow{d_{n+1}} & \cdots
\end{array} \tag{3.2}$$

Before we move on to other induced categories, let's consider here some properties of complexes and chain maps. For a given complex $\mathcal{E}^\bullet : \cdots \xrightarrow{c_{n-1}} \mathcal{E}^n \xrightarrow{c_n} \mathcal{E}^{n+1} \xrightarrow{c_{n+1}} \cdots$, one is tempted to define its cohomology as usual:

$$\mathrm{H}^n(\mathcal{E}^\bullet) = \operatorname{Ker} c_n / \operatorname{Im} c_{n-1} \tag{3.3}$$

For a general Abelian category, $\mathbb{A}$, a more subtle procedure is needed to define the cohomology, as the usual notion of *"modding out"* the objects does not make sense in a general Abelian category. However, for the abelian categories we are interested in, the usual notions of kernels and cokernels make sense, and we can define the cohomology of the complex in the straightforward fashion (3.3).

Note that cohomology $\mathrm{H}^\bullet(\mathcal{E}^\bullet)$ itself can be regarded as a complex:

$$\cdots \xrightarrow{0} \mathrm{H}^n(\mathcal{E}^\bullet) \xrightarrow{0} \mathrm{H}^{n+1}(\mathcal{E}^\bullet) \xrightarrow{0} \cdots \tag{3.4}$$

Then, one can easily verify that a chain map $f : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ induces another chain map $\mathrm{H}(f) : \mathrm{H}^\bullet(\mathcal{E}^\bullet) \to \mathrm{H}^\bullet(\mathcal{F}^\bullet)$. If $\mathrm{H}(f)$ is an isomorphism, the chain map f is said to be a *quasi-isomorphism.*

Here, let us take a moment to explain why these complicated mathematical concepts like complexes and quasi-isomorphisms play important roles in describing D-branes. First, in [3, 19, 23], it is shown that a subclass of B-type D-branes on a Calabi-Yau X – namely those which correspond to branes in the B-twisted topological string theory – can be described as complexes of coherent sheaves on X. It turns out that quasi-isomorphic complexes lead to identical open string spectra. So we would like to identify them as being "isomorphic". But, in $\mathrm{Kom}(\mathbb{A})$, quasi-isomorphisms are *not* invertible. We need to pass to some fancied-up version, the *derived category*, $D(\mathbb{A})$, in which quasi-isomorphisms are turned into isomorphisms. The objects of $D(\mathbb{A})$ are complexes, as before. Just the morphisms have been changed.

But now an added payoff emerges, given a pair of complexes, $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$, the space of morphisms in the derived category from $\mathcal{E}^\bullet$ to $\mathcal{F}^\bullet$, $\mathrm{Hom}_{D(\mathbb{A})}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is isomorphic to the Hilbert space of states of the topological open string stretched between the corresponding D-branes (the space of *chiral* operators in the corresponding physical open string). This is consistent with the above remark because isomorphic objects have the same (isomorphic) spaces of morphisms to (from) any other object.

The procedure of inverting is similar to that of making abelian semi-groups into groups. For example, the number 4 does not have an inverse in the additive semi-group $\mathbb{Z}^{\geq 0} \equiv \mathbb{N} \cup \{0\}$. But, we can invert it by *"creating"* its inverse -4. Inverting all positive integers in this way, we extend our semi-group to a group, $\mathbb{Z}$.

In the same way, we are going to extend our morphisms in $\mathrm{Kom}(\mathbb{A})$ by creating inverses of all quasi-isomorphisms. Doing so, we will end up with a new category where quasi-isomorphisms in $\mathrm{Kom}(\mathbb{A})$ have their inverses and hence are isomorphisms. This new category is called the *derived category.* However, the procedure of inverting quasi-isomorphisms is not so simple as in our abelian semi-group example. This is because composing two morphisms is not commutative while the semi-group action was. To resolve this difficulty, we need to introduce, as an intermediate step, another category called the *homotopy category*, $K(\mathbb{A})$.

Objects of $K(\mathbb{A})$ are the same as those of $\mathrm{Kom}(\mathbb{A})$ but the morphisms are different. Two chain maps $f, f' : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ are said to be *homotopic* if there exists a family of morphisms $h^n \in \mathrm{Hom}_{\mathbb{A}}(\mathcal{E}^n, \mathcal{F}^{n-1})$ such that $f'^n = f^n + h^{n+1} \circ c^n + d^{n-1} \circ h^n$.

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{E}^{n-1} \arrow[r, "c^{n-1}"] \arrow[d, shift right, "f^{n-1}"'] \arrow[d, shift left, dashed, "f'^{n-1}"] & \mathcal{E}^n \arrow[r, "c^n"] \arrow[dl, "h^n"'] \arrow[d, shift right, "f^n"'] \arrow[d, shift left, dashed, "f'^n"] & \mathcal{E}^{n+1} \arrow[r] \arrow[dl, "h^{n+1}"'] \arrow[d, shift right, "f^{n+1}"'] \arrow[d, shift left, dashed, "f'^{n+1}"] & \cdots \\
\cdots \arrow[r] & \mathcal{F}^{n-1} \arrow[r, "d^{n-1}"] & \mathcal{F}^n \arrow[r, "d^n"] & \mathcal{F}^{n+1} \arrow[r] & \cdots
\end{tikzcd} \tag{3.5}$$

The morphisms of $K(\mathbb{A})$ are morphisms of $\mathrm{Kom}(\mathbb{A})$ modulo homotopy equivalence. Since homotopic maps induce the same map on cohomology – $\mathrm{H}(f') = \mathrm{H}(f)$ if f is homotopic to f' – we have the same notion of quasi-isomorphisms as before. But $K(\mathbb{A})$ has a very special property regarding quasi-isomorphisms. Namely, the class of quasi-isomorphisms is *localizing* in $K(\mathbb{A})$. We will not give the rigorous definition of localization here (see [28]). But we will make use of its consequences.

A *roof* from complex $\mathcal{E}^\bullet$ to complex $\mathcal{F}^\bullet$ is a diagram (s, f) of the form:

$$\begin{tikzcd}
& \mathcal{G}^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & \\
\mathcal{E}^\bullet & & \mathcal{F}^\bullet
\end{tikzcd} \tag{3.6}$$

where s is a quasi-isomorphism in $\mathrm{Hom}_{K(\mathbb{A})}(\mathcal{G}^\bullet, \mathcal{E}^\bullet)$ and $f \in \mathrm{Hom}_{K(\mathbb{A})}(\mathcal{G}^\bullet, \mathcal{F}^\bullet)$. Two roofs are equivalent, $(s, f) \sim (t, g)$, if and only if there exists a third roof forming a commutative diagram of the form:

$$\begin{tikzcd}
 & & \mathcal{I}^\bullet \arrow[dl, dashed, "r"'] \arrow[dr, dashed, "h"] & & \\
 & \mathcal{G}^\bullet \arrow[dl, "s"'] \arrow[drrr, "f"' near end] & & \mathcal{H}^\bullet \arrow[dr, "g"] \arrow[dlll, "t" near end] & \\
\mathcal{E}^\bullet & & & & \mathcal{F}^\bullet
\end{tikzcd} \tag{3.7}$$

One of the nice consequences of localization is the following. Suppose we have two roofs, $(s, f) : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ and $(t, g) : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$. Then there exists a third roof (r, h) making the following diagram commutative:

$$\begin{tikzcd}
 & & \mathcal{J}^\bullet \arrow[dl, dashed, "r"'] \arrow[dr, dashed, "h"] & & \\
 & \mathcal{H}^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & & \mathcal{I}^\bullet \arrow[dl, "t"'] \arrow[dr, "g"] & \\
\mathcal{E}^\bullet & & \mathcal{F}^\bullet & & \mathcal{G}^\bullet
\end{tikzcd} \tag{3.8}$$

Note that in the above diagram, $(s \circ r, g \circ h)$ is also a roof. This localizing property enables one to define the composition of two equivalence classes of roofs. That is

$$[s, f] \circ [t, g] \equiv [s \circ r, g \circ h]. \tag{3.9}$$

Now we have all the technology to define the *derived category* $D(\mathbb{A})$.

- Objects of $D(\mathbb{A})$ are complexes in $\mathbb{A}$.

- Morphisms of $D(\mathbb{A})$ are equivalence classes of roofs in $K(\mathbb{A})$.

- The composition law of morphisms is defined as in (3.8), (3.9).

- The identity morphism $id_{\mathcal{E}^\bullet}$ is $[id_{\mathcal{E}^\bullet}, id_{\mathcal{E}^\bullet}]$.

By defining morphisms of $D(\mathbb{A})$ as above, we effectively *"invert"* quasi-isomorphisms. The inverse of quasi-isomorphism $[id, s]$ is $[s, id]$. Note $[id, s] \circ [s, id] = [s, id] \circ [id, s] = [s, s] \sim [id, id]$ since the following diagram is commuting:

$$\begin{tikzcd}
& & \mathcal{E}^\bullet \arrow[dl, dashed, "id"'] \arrow[dr, dashed, "s"] & & \\
& \mathcal{E}^\bullet \arrow[dl, "s"'] \arrow[rrrd, "s"] & & \mathcal{F}^\bullet \arrow[llld, "id"'] \arrow[dr, "id"] & \\
\mathcal{F}^\bullet & & & & \mathcal{F}^\bullet
\end{tikzcd} \tag{3.10}$$

We had to pass to the homotopy category $K(\mathbb{A})$ in order to define our roofs. If we'd tried to define them in the category of complexes, $Kom(\mathbb{A})$, the composition of two roofs would not have been equivalent to another roof. Physically, imposing equivalence of chain maps up to homotopy is very closely related to the BRST construction of [19].

There are fancier and more abstract definitions of the derived category (see [28]), but this one is *constructive*, and allows you to actually compute the space of morphisms (and hence the spectrum of chiral operators of the open string theory) .

Before finishing this subsection, let's introduce some useful notations. As mentioned earlier, $\mathbb{A}$ will be the category of coherent sheaves on a Calabi-Yau throughout the paper. Hence it will be useful to denote the (bounded) derived category of coherent sheaves on X by $D(X)$. Also, we will frequently encounter complexes that consist of one entry. For example:

$$\cdots \to 0 \to \mathcal{E} \to 0 \to \cdots \tag{3.11}$$

We will denote the above complex by $\mathcal{E}[n]$ where n indicates $\mathcal{E}$ is at the $-n$ position in the complex. The position of the sheaves in the complex is usually called the *grading* in physics literatures. Sometimes, we will put small numbers above the sheaves in a complex to denote their positions. For example:

$$\cdots \to \overset{-4}{\mathcal{E}} \to \overset{-3}{\mathcal{F}} \to \overset{-2}{\mathcal{G}} \to \cdots \tag{3.12}$$

Finally, there is an obvious functor from $D(X)$ to itself called the shift functor. It simply shifts the grading on all the objects (and morphisms). The complex $\mathcal{E}^\bullet[n]$ denotes the complex $\mathcal{E}^\bullet$ shifted n places to the left.

3.1.2 Object simplification

Two complexes, $\mathcal{E}_1^\bullet$ and $\mathcal{E}_2^\bullet$, which are quasi-isomorphic lead to open string theories with the same spectrum (not just for the strings beginning and ending on $\mathcal{E}_i^\bullet$, but also for the strings stretched between $\mathcal{E}_i$ and and any other brane, $\mathcal{F}^\bullet$). Thus we should identify $\mathcal{E}_1^\bullet$ and $\mathcal{E}_2^\bullet$ as representing the "same" D-brane. We then might wish to ask to what extent we can use quasi-isomorphisms to replace a given complex by a "simpler" one.

Consider the complex

$$0 \to \mathcal{E}^0 \xrightarrow{c_0} \mathcal{E}^1 \xrightarrow{c_1} \mathcal{E}^2 \to \cdots \to \mathcal{E}^{N-1} \xrightarrow{c_{N-1}} \mathcal{E}^N \to 0 \tag{3.13}$$

Let us assume that (3.13) is exact at the $\mathcal{E}^0$ term (*i.e.*, that c_0 is injective). Then there is a quasi-isomorphism

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}^0 \arrow[r, "c_0"] & \mathcal{E}^1 \arrow[r, "c_1"] \arrow[d, "q"'] & \mathcal{E}^2 \arrow[r, "c_2"] \arrow[d, "1"'] & \cdots \\
& 0 \arrow[r] & \mathcal{E}^1/c_0(\mathcal{E}^0) \arrow[r, "c_1"] & \mathcal{E}^2 \arrow[r, "c_2"] & \cdots
\end{tikzcd} \tag{3.14}$$

So we can replace (3.13) by the simpler complex

$$0 \to \mathcal{E}^1/c_0(\mathcal{E}^0) \xrightarrow{c_1} \mathcal{E}^2 \xrightarrow{c_2} \cdots$$

By iterating this process, we can always eliminate all the exact terms on the left-hand side of the complex.

Conversely, consider the case where (3.13) is exact at the $\mathcal{E}^N$ term (*i.e.* that c_{N-1} is surjective). Then we can find a quasi-isomorphism

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{E}^{N-2} \arrow[r, "c_{N-2}"] & \mathcal{E}^{N-1} \arrow[r, "c_{N-1}"] & \mathcal{E}^N \arrow[r] & 0 \\
\cdots \arrow[r] & \mathcal{E}^{N-2} \arrow[r, "c_{N-2}"] \arrow[u, "1"] & \ker(c_{N-1}) \arrow[r] \arrow[u, "i"] & 0 &
\end{tikzcd} \tag{3.15}$$

Again, by iterating this process, we can eliminate all the exact terms on the right of the complex.

Combining the two operations, we can, without loss of generality, assume that there is cohomology in the first nonzero term in the complex and in the last nonzero term. If our complex is a direct sum of complexes, $\mathcal{E}^\bullet = \mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet$, then we can apply this procedure to each of the direct summands separately.

But, in general, if the cohomology occurs in more than one term of the complex, we cannot simplify further. In particular, we typically *cannot* reduce a complex of coherent sheaves, $\mathcal{E}^\bullet$, to its cohomology for they are usually *not* quasi-isomorphic.

As a simple (indeed, the prototypical) example of this, let $X = T^2 \times T^2$, with coordinates (z_1, z_2) and let the divisors $D_i = \{z_i = 0\}$.

The complex

$$0 \to \mathcal{O}(-D_1) \xrightarrow{z_1} \mathcal{O} \to 0 \tag{3.16}$$

has cohomology only in the second term and, indeed, is quasi-isomorphic to

$$0 \to 0 \to \mathcal{O}_{D_1} \to 0 \tag{3.17}$$

where $\mathcal{O}_{D_1}$ is the structure sheaf of the divisor D_1 (extended by zero to a coherent sheaf on X). Physically, a wrapped D4-brane and a wrapped anti-D4 brane (which carries one unit of D2-brane charge on its worldvolume) can annihilate into a D2-brane wrapped on D_1 via tachyon condensation and (3.16) and (3.17) are different ways of expressing the endpoint of that condensation.

So far so good. But now consider the complex

$$0 \to \mathcal{O}(-D_1) \oplus \mathcal{O}(-D_2) \xrightarrow{(\, z_1 \ z_2 \,)} \mathcal{O} \to 0 \tag{3.18}$$

The cohomology of this complex is

$$0 \to \mathcal{O}(-D_1 - D_2) \to \mathcal{O}_p \to 0 \tag{3.19}$$

where $p = D_1 \cap D_2 = \{z_1 = z_2 = 0\}$. But (3.18) and (3.19) are *not* quasi-isomorphic.

3.1.3 The intersection pairing

In [11, 12], one of the guiding principles was that the monodromies acting on the K-theory should preserve the skew-symmetric bilinear pairing on $K(X)$, which, in the Calabi-Yau context, can be written as

$$(v, w) = \int_X ch(v \otimes \overline{w}) Td(X) \tag{3.20}$$

This pairing can be written for objects in the derived category as[a]

$$(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \sum_i (-1)^i \dim \mathrm{Hom}_{D(X)}(\mathcal{F}^\bullet, \mathcal{E}[i]^\bullet) \tag{3.21}$$

The sum on i is always a finite one because we are working with the *bounded* derived category. Any functor $F : D(X) \to D(X)$ will automatically preserve (3.21), provided

a) F is fully faithful[b].

b) F commutes, up to quasi-isomorphism, with the shift functor $[n]$.

This will certainly hold for the monodromies defined by the kernels in §3.2,§3.3.

---

[a]For a pair of coherent sheaves, $\mathrm{Hom}_{D(X)}(\mathcal{F}[0], \mathcal{E}[i]) = \mathrm{Ext}^i(\mathcal{F}, \mathcal{E})$, so (3.21) reduces to

$$(\mathcal{E}[0], \mathcal{F}[0]) = \sum_i (-1)^i \dim \mathrm{Ext}^i(\mathcal{F}, \mathcal{E})$$

[b]A functor $F : \mathbb{A} \to \mathbb{B}$ of additive categories is *fully faithful* if, for any $X, Y \in Ob(\mathbb{A})$,

$$F : \mathrm{Hom}_{\mathbb{A}}(X, Y) \to \mathrm{Hom}_{\mathbb{B}}(F(X), F(Y))$$

is an isomorphism.

3.1.4 Quantum effects

We will end this "review" section with some comments on how some of the information (specifically, the grading) in the derived category can be wiped out by quantum string effects (as we will argue in §3.3.3, the grading is already only well-defined modulo 6).

Consider the pair of objects

$$\mathcal{E}^\bullet \;=\; 0 \to \mathcal{O} \to 0 \to \mathcal{O} \longrightarrow 0 \;\simeq\; 0 \to \mathcal{O} \to \mathcal{O} \xrightarrow{\binom{0}{1}} \mathcal{O}^{\oplus 2} \to 0$$

and

$$\mathcal{F}^\bullet \;=\; 0 \to 0 \to 0 \to \mathcal{O}^{\oplus 2} \to 0 \;\simeq\; 0 \to \mathcal{O} \xrightarrow{1} \mathcal{O} \longrightarrow \mathcal{O}^{\oplus 2} \to 0$$

$\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ are clearly not quasi-isomorphic (they're equal to their cohomologies, which are clearly not isomorphic). The open string CFT on $\mathcal{F}^\bullet$ has 4 chiral primaries (which give rise to a $U(2)$ gauge theory on its world-volume), whereas the open string CFT on $\mathcal{E}^\bullet$ has only 2 (giving rise to a $U(1) \times U(1)$ gauge theory).

However, both of these branes can be viewed as the endpoints of tachyon condensation of the unstable brane

$$\mathcal{G}^\bullet \;\simeq\; 0 \to \mathcal{O} \to \mathcal{O} \to \mathcal{O}^{\oplus 2} \to 0$$

This has a $U(2) \times U(1) \times U(1)$ gauge theory and, at a typical point in the large-radius regime, can decay to *either* $\mathcal{E}^\bullet$ or $\mathcal{F}^\bullet$ by condensing *different* tachyons.

At least in the large-radius regime, one expects that $\mathcal{E}^\bullet$ is actually an excited state of $\mathcal{F}^\bullet$. But there isn't an obvious tachyon that one can condense. One expects, instead, that the decay of $\mathcal{E}^\bullet$ to $\mathcal{F}^\bullet$ proceeds via barrier-penetration in the world-volume gauge theory. This is an effect not readily visible in the perturbative open-string theory.

More generally, consider a D-brane corresponding to some complex of coherent sheaves, at a *particular point* in the Kähler moduli space. It is, in general, not easy to decide whether this brane is stable (BPS) at this point in the moduli space. One must check the π-stability criterion of [23, 25] for every distinguished triangle in which this complex participates. This is not an easy task.

So, in the following, we will blithely talk about the monodromy mapping *this* complex of coherent sheaves into *that* complex. But we will not make any claim that the resulting complex corresponds to a *stable* D-brane in the large-radius regime.

3.2 Monodromies

Acting on the derived category, the monodromies are generated by kernels $K^\bullet \in D^b(X \times X)$. The formula for the monodromy action on a complex $\mathcal{E}^\bullet$ is

$$\mathcal{E}^\bullet \mapsto Rp_{1*}(K^\bullet \overset{L}{\otimes} p_2^*(\mathcal{E}^\bullet)) \tag{3.22}$$

The name "kernel" is deliberately chosen to remind you that this transformation is closely analogous to an integral transform,

$$f(x) \mapsto \int dy\ K(x,y) f(y)$$

Indeed, many of the formulæ one writes for integral transforms (*e.g.* for the composition of two such transforms) have precise analogues here. In (3.22), we

1. Take a complex of sheaves $\mathcal{E}^\bullet$ on X, "pull it back" to the inverse-image complex of sheaves, $p_2^*(\mathcal{E}^\bullet)$ on $X \times X$.

2. Take the tensor-product with the kernel, $K^\bullet$, and construct the left-derived complex of sheaves.

3. Finally, we "push-forward" to the direct image complex, $p_{1*}(\cdot)$, and construct the right-derived complex of sheaves on X.

Each of these steps sounds a little formidable, but, in practise, they are not. The left-derived functor, $\cdot \overset{L}{\otimes} \mathcal{F}^\bullet$ is constructed by taking a complex, $V^\bullet$, of locally-free sheaves which is quasi-isomorphic to $\mathcal{F}^\bullet$, and computing the *ordinary* tensor-product with $V^\bullet$. So, in step 1, we replace $\mathcal{E}^\bullet$ by a quasi-isomorphic complex of locally-free sheaves (sheaves of sections of holomorphic vector bundles V^n). The inverse-image of this complex is the complex of sheaves of sections of the pullback bundles $p_2^* V^n$. This is again locally-free, so we can take the ordinary tensor product with $K^\bullet$.

In step 3, we need to take the direct-image. Unfortunately, this step is more complicated than the previous ones. Let's consider simpler case, first.

$$p : X \to \mathrm{pt} \tag{3.23}$$

Say we have a complex, $\mathcal{E}^\bullet$, of sheaves on X. $Rp_*\mathcal{E}^\bullet$ is a complex of sheaves over a point, *i.e.* a complex of vector spaces. (We remind the reader that a complex of vector spaces is quasi-isomorphic to its cohomology.) The raw ingredients for computing $Rp_*\mathcal{E}^\bullet$ are given by the groups $\mathrm{H}^i(X, \mathcal{E}^j)$. But the relation is subtle. What one must compute is the spectral sequence of a double complex, whose E_1 term is

$$E_1^{p,q} = \mathrm{H}^q(X, \mathcal{E}^p) \tag{3.24a}$$

and whose first differential

$$d_1 : \mathrm{H}^q(X, \mathcal{E}^p) \to \mathrm{H}^q(X, \mathcal{E}^{p+1}) \tag{3.24b}$$

is the map on cohomology induced from the differential of the original complex. This spectral sequence is well-known, and converges to the *hypercohomology*[c] of the complex $\mathcal{E}^\bullet$ (see [35], p. 445):

$$E_\infty^{p,q} \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{E}^\bullet) \tag{3.24c}$$

[c] In the Čech model for sheaf cohomology, we have a Čech coboundary operator, $\delta : C^n(\mathcal{E}) \to C^{n+1}(\mathcal{E})$. For a complex, $\mathcal{E}^\bullet$, of sheaves, we also have the differential of the complex, $d : \mathcal{E}^n \to \mathcal{E}^{n+1}$. These satisfy $\delta^2 = d^2 = d\delta + \delta d = 0$. The hypercohomology $\mathbb{H}^n(X, \mathcal{E}^\bullet)$ is the cohomology of $D = d + \delta$ acting on the total complex.

So, in the case of (3.23), $Rp_*\mathcal{E}^\bullet$ is a complex whose differentials all vanish, and whose n^{th} entry is

$$(Rp_*\mathcal{E}^\bullet)^n = \mathbb{H}^n(X, \mathcal{E}^\bullet) \tag{3.25}$$

It is a theorem proven in [35] that a quasi-isomorphism $s : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ induces an *isomorphism* on the hypercohomology $\hat{s} : \mathbb{H}^\bullet(X, \mathcal{E}^\bullet) \xrightarrow{\sim} \mathbb{H}^\bullet(X, \mathcal{F}^\bullet)$. Hence (3.25) has the desired property that it take quasi-isomorphic complexes into quasi-isomorphic complexes.

Because of this property, we can use any complex $\mathcal{F}^\bullet$ quasi-isomorphic to $\mathcal{E}^\bullet$ to calculate the hypercohomolgy. In particular, we want to choose a complex whose hypercohomology is simple enough to calculate. One nice choice is a complex made out of sheaves $\mathcal{F}^n$ whose higher cohomologies vanish, $\mathrm{H}^i(X, \mathcal{F}^n)$ for $i > 0$[d]. Hypercohomology of such a complex $\mathcal{F}^\bullet$ is simply the cohomology of complex, $p_*(\mathcal{F}^\bullet)$.

Now, let's generalize the above discussion to the case at hand. In our case, we have

$$p_1 : X \times X \to X. \tag{3.26}$$

If we fix a point p in the first factor of $X \times X$, this problem reduces to the previous one. So, basicaly, what we have to do is to repeat the procedure above for each $p \in X$. We replace the original complex $\mathcal{E}^\bullet$ with a quasi-isomorphic

[d]Such sheaves form a so-called *adapted class* to functor p_*

complex $\mathcal{F}^\bullet$ that has the following nice property[e]:

$$\mathrm{H}^i(X, \mathcal{F}^n|_{\{p\}\times X}) = 0 \qquad \text{for all } i > 0 \text{ and } p \in X \tag{3.27}$$

Here, $\mathcal{F}^n|_{\{p\}\times X}$ is a sheaf on X constructed from $\mathcal{F}^n$ using the *stalk-like construction.* The image of $\mathcal{F}^\bullet$ under Rp_{1*} is simply $p_{1*}(\mathcal{F}^\bullet)$ as in the previous example. There is a subtlety here. In the previous example, we were allowed to take the cohomology of the image $p_*(\mathcal{F}^\bullet)$ to get the hypercohomogy since every complex of vector spaces is quasi-isomorphic to its cohomology. That is not true any more in our case. We can't take the cohomology of the image, $p_{1*}(\mathcal{F}^\bullet)$.

To end this subsection, let us summarize the procedure of the monodromy calculations for given complex $\mathcal{E}^\bullet$.

1. Replace $\mathcal{E}^\bullet$ with a quasi-isomorphic complex $V^\bullet$ of locally free sheaves.
2. Take inverse image $p_2^*(V^\bullet)$.
3. Take the ordinary tensor product with $K^\bullet$ term by term.
4. Replace the above complex with a quasi-isomorphic complex with property (3.27).
5. Take the oridnary direct image.

[e]Again, the collection of sheaves with the above property is a *adapted class* to functor p_{1*}.

This procedure is for general kernel $K^\bullet$. For the most of kernels we will consider in this paper, the procudure is going to be simplified a lot due to the specific forms of kernels.

3.2.1 Some kernels

The first, and most obvious monodromy is that about the large-radius limit (shifting the B-field by an integral class $\xi \in \mathrm{H}^2(X, \mathbb{Z})$). This acts on the D-branes by tensoring them with a line bundle L, with $c_1(L) = \xi$. In terms of kernels,

$$K_r^\bullet = (j_* L)[0] \tag{3.28}$$

This action is a simple application of the push-pull formula

$$\begin{aligned} M_r(\mathcal{E}^\bullet) &= Rp_{1*}((j_* L)[0] \overset{L}{\otimes} p_2^*(\mathcal{E}^\bullet)) \\ &= Rp_{1*}(j_*(L \otimes j^* p_2^*(\mathcal{E}^\bullet))) \\ &= Rp_{1*}(j_*(L \otimes (p_2 \circ j)^*(\mathcal{E}^\bullet))) \\ &= Rp_{1*}(j_*(L \otimes \mathcal{E})^\bullet) \\ &= R(p_1 \circ j)_*((L \otimes \mathcal{E})^\bullet) \\ &= (L \otimes \mathcal{E})^\bullet \end{aligned}$$

That is, we tensor the complex $\mathcal{E}^\bullet$ term-by-term with the line bundle L.

The next, most obvious monodromy is that about the (mirror of the) conifold (the principal component of the discriminant locus). Kontsevich [41] conjectured that this was given by

$$K_c^\bullet \overset{?}{=} 0 \to \overset{-1}{\mathcal{O}_{X\times X}} \xrightarrow{r} \overset{0}{\mathcal{O}_\Delta} \to 0 \tag{3.29}$$

where r is the restriction to the diagonal and where we have indicated by superscripts the grades of the sheaves in the complex. We will propose two modifications of this formula. First, we need to compose Kontsevich's kernel with the "shift-by-two" functor.

$$K_c^\bullet = 0 \to \overset{-3}{\mathcal{O}_{X\times X}} \xrightarrow{r} \overset{-2}{\mathcal{O}_\Delta} \to 0 \qquad (3.30)$$

The reason is that we want $M_c(\mathcal{O}[0]) = \mathcal{O}[0]$. Physically, the D6-brane becomes massless at the conifold locus, and there is a local description of the physics near the conifold locus when it is included as a fundamental field in the 4D action. Thus it must be single-valued with respect to M_c. Had we used (3.29) instead, we would have gotten $M_c(\mathcal{O}[k]) = \mathcal{O}[k-2]$ [f].

Our second modification is required to fix (3.30) on nonsimply-connected Calabi-Yau manifolds. If the holonomy of X is $SU(3)$ and not a proper subgroup, then the fundamental group $\pi_1(X)$ is finite. We can associate a flat holomorphic vector bundle, W_j to each irreducible representation of $\pi_1(X)$. We replace (3.30) with

$$K_c^\bullet = 0 \to \overset{-3}{\bigoplus_j W_j \boxtimes W_j^*} \xrightarrow{\tilde{r}} \overset{-2}{\mathcal{O}_\Delta} \to 0 \qquad (3.31)$$

where $\tilde{r}$ is restriction to the diagonal, followed by taking the trace. This sum over irreducible representations of $\pi_1(X)$ will induce the correct action on the

[f] After completing this paper, it came to our attention that some of the monodromy calculations of this and the next section are done in [2], but using (3.29) instead of (3.30).

K-theory, as computed in [12], namely

$$v \mapsto v - \sum_j (v, W_j) W_j \tag{3.32}$$

where v is the K-theory class of the D-brane in question, and $(\cdot,\cdot)$ is the skew-symmetric bilinear form (3.20) on $K^0(X)$.

With kernel (3.31)(or its specialization to simply-connected Calabi-Yau, (3.30)), the monodromy calculation will be simpler than the general case. First, it is easy to show that, with this specific kernel, $K_c^\bullet \otimes p_2^* \cdot$ is an exact functor. Thus, its left derived functor, $K_c^\bullet \overset{L}{\otimes} p_2^* \cdot$ is the functor itself. For a given complex $\mathcal{E}^\bullet$, we do not need to replace it with a quasi-isomorphic complex of locally free sheaves.

Further simplification can be made if we replace $\mathcal{E}^\bullet$ with a quasi-isomorphic complex $\mathcal{F}^\bullet$ with the following property[g]:

$$\mathrm{H}^i(X, \mathcal{F}^n_{-j}) = 0 \qquad\qquad \text{for all } i > 0,\ n, \text{ and } j \tag{3.33}$$

where we use $\mathcal{F}^n_{-j}$ to denote $W_j^* \otimes \mathcal{F}^n$. A nice thing about such a complex $\mathcal{F}^\bullet$ is that $K_c^\bullet \otimes p_2^* \mathcal{F}^\bullet$

$$\begin{gathered}\cdots \xrightarrow{\begin{pmatrix} 1\boxtimes d & 0 \\ (-)^{n-4}\tilde{r} & d \end{pmatrix}} \overset{n-3}{\bigoplus_j W_j \boxtimes \mathcal{F}^n_{-j} \oplus \mathcal{F}^{n-1}_\Delta} \\ \xrightarrow{\begin{pmatrix} 1\boxtimes d & 0 \\ (-)^{n-3}\tilde{r} & d \end{pmatrix}} \overset{n-2}{\bigoplus_j W_j \boxtimes \mathcal{F}^{n+1}_{-j} \oplus \mathcal{F}^n_\Delta} \xrightarrow{\begin{pmatrix} 1\boxtimes d & 0 \\ (-)^{n-2}\tilde{r} & d \end{pmatrix}} \cdots\end{gathered} \tag{3.34}$$

[g]For example, one can use injective resolutions. If I is injective, so is $V \otimes I$ for any locally free sheaf V. Hence, $\mathrm{H}^i(X, I_{-j}) = 0$ for $i > 0$

automatically satisfies the condition (3.27). The calculation of Rp_{1*} becomes easy. We just take the image under p_{1*}:

$$
\begin{aligned}
\cdots \xrightarrow{\begin{pmatrix} 1\otimes\Gamma(d) & 0 \\ (-)^{n-4}\tilde{ev} & d \end{pmatrix}} & \overset{n-3}{\bigoplus_j W_j \otimes \Gamma(\mathcal{F}^n_{-j}) \oplus \mathcal{F}^{n-1}} \\
& \xrightarrow{\begin{pmatrix} 1\otimes\Gamma(d) & 0 \\ (-)^{n-3}\tilde{ev} & d \end{pmatrix}} \overset{n-2}{\bigoplus_j W_j \otimes \Gamma(\mathcal{F}^{n+1}_{-j}) \oplus \mathcal{F}^{n}} \xrightarrow{\begin{pmatrix} 1\otimes\Gamma(d) & 0 \\ (-)^{n-2}\tilde{ev} & d \end{pmatrix}} \cdots
\end{aligned}
\tag{3.35}
$$

with $\tilde{ev} = \Gamma(\tilde{r})$, *i.e.* evaluation followed by taking the trace.

This is not the end of the story yet. One can simplify it further. $\Gamma(\mathcal{F}^n_{-j})$ is just a plain vector space and hence, complex $\Gamma(\mathcal{F}^{\bullet}_{-j})$ is quasi-isomorphic to its cohomology. Let H^n_{-j} be the nth entry of this cohomology,

$$
\mathrm{Ker}\{\Gamma(d) : \Gamma(\mathcal{F}^n_{-j}) \to \Gamma(\mathcal{F}^{n+1}_{-j})\}/\mathrm{Im}\{\Gamma(d) : \Gamma(\mathcal{F}^{n-1}_{-j}) \to \Gamma(\mathcal{F}^n_{-j})\}. \tag{3.36}
$$

Then, it is not too difficult to show that the complex above is quasi-isomorphic to

$$
\begin{aligned}
\cdots \xrightarrow{\begin{pmatrix} 0 & 0 \\ (-)^{n-4}\tilde{ev} & d \end{pmatrix}} & \overset{n-3}{\bigoplus_j W_j \otimes \mathrm{H}^n_{-j} \oplus \mathcal{F}^{n-1}} \\
& \xrightarrow{\begin{pmatrix} 0 & 0 \\ (-)^{n-3}\tilde{ev} & d \end{pmatrix}} \overset{n-2}{\bigoplus_j W_j \otimes \mathrm{H}^{n+1}_{-j} \oplus \mathcal{F}^{n}} \xrightarrow{\begin{pmatrix} 0 & 0 \\ (-)^{n-2}\tilde{ev} & d \end{pmatrix}} \cdots .
\end{aligned}
\tag{3.37}
$$

Generally, this is as simple as it gets. Using the oject simplification method we have discussed, one may go further by eliminating all the exact terms on the left- and right- hand sides of the complex. In this way, we get a shorter complex, but the complex itself would look more complicated.

Let's apply the discussion above to some examples. The first example we will look at is $W_k[0]$. For the Calabi-Yau spaces we will study, it is fairly easy to prove that the sheaf cohomology groups, $\mathrm{H}^i(X, W_j) = 0$ except for the trivial representation, $W_j = \mathcal{O}$. Similarly, we have

$$W_j^* \otimes W_k = \begin{cases} \mathcal{O} \oplus \cdots & j = k \\ \cdots & \text{otherwise} \end{cases} \tag{3.38}$$

where "$\cdots$" denote flat bundles associated to nontrivial irreps of $\pi_1(X)$. Let $\mathcal{F}^\bullet$ be a complex quasi-isomorphic to $W_k[0]$ with property (3.33):

$$\begin{tikzcd} 0 \arrow[r] & W_k \arrow[r] \arrow[d, "f"'] & 0 \\ 0 \arrow[r] & \mathcal{F}^0 \arrow[r, "d^0"] & \mathcal{F}^1 \arrow[r, "d^1"] & \cdots . \end{tikzcd} \tag{3.39}$$

To apply (3.37), we need to calculate the cohomology of complex $\Gamma(\mathcal{F}^\bullet_{-j})$. One way to do it is to use the exact sequence $0 \to W_k \xrightarrow{f} \mathcal{F}^\bullet$. Since tensoring with a locally free sheaf is an exact functor,

$$0 \to W_j^* \otimes W_k \xrightarrow{1\otimes f} \mathcal{F}^0_{-j} \xrightarrow{1\otimes d^0} \mathcal{F}^1_{-j} \xrightarrow{1\otimes d^1} \cdots \tag{3.40}$$

is also exact for all j. $\mathcal{F}^n_{-j}$ does not have any higher cohomology due to the choice of $\mathcal{F}^\bullet$. Hence, the above exact sequence implies that the cohomology of the complex $\Gamma(\mathcal{F}^\bullet_{-j})$ is given by the sheaf cohomology $\mathrm{H}^\bullet(X, W_j^* \otimes W_k)$. They all vanish unless $j = k$ and when $j = k$, the cohomology of the complex is

$$0 \to \overset{0}{\mathbb{C}} \to 0 \to 0 \to \overset{3}{\mathbb{C}} \to 0. \tag{3.41}$$

Combining all those facts, now we calcualte $M_c(W_k[0])$:

$$\begin{aligned} M_c(W_k[0]) &\simeq 0 \to \overset{-3}{W_k} \xrightarrow{-f} \overset{-2}{\mathcal{F}^0} \xrightarrow{d^0} \overset{-1}{\mathcal{F}^1} \xrightarrow{\binom{0}{d^1}} \overset{0}{W_k \oplus \mathcal{F}^2} \xrightarrow{(\tilde{ev}\ d^2)} \overset{1}{\mathcal{F}^3} \xrightarrow{d^3} \overset{2}{\mathcal{F}^4} \xrightarrow{d^4} \cdots \\ &\simeq W_k[0]. \end{aligned} \tag{3.42}$$

This is what we expect, since, as we argued in [12], all of the 6-branes corresponding to the W_j become massless at the conifold. Hence they all must be single-valued under M_c.

For another example, let's consider a D0-brane. A D0-brane sitting at the point p is represented by the skyscraper sheaf $\mathcal{O}_p[0]$. The higher cohomology of $W_j^* \otimes \mathcal{O}_p$ vanishes. So, we do not need to make any replacement here. Let's compute what happens to the D0-brane when we circle the conifold.

$$\begin{aligned} M_c(\mathcal{O}_p[0]) &= 0 \to \overset{-3}{\bigoplus_j W_j \otimes \Gamma(W_j^* \otimes \mathcal{O}_p)} \xrightarrow{\tilde{ev}} \overset{-2}{\mathcal{O}_p} \to 0 \\ &\simeq \left({\bigoplus_j}' W_j \oplus \mathcal{I}_p \right) [3] \end{aligned} \tag{3.43}$$

where $\mathcal{I}_p$ is the *ideal sheaf* of the point p and the direct sum $\bigoplus_j'$ runs over nontrivial irreps of $\pi_1(X)$. So, the D0-brane has turned into a collection of anti-D6-branes and another anti-D6-brane with one unit of D0-brane charge dissolved on it.

3.3 The Quintic and its Orbifold

To obtain further concrete results, we need to specialize to some examples. We will look at D-branes on Y, the quintic hypersurface in $\mathbb{P}^4$, and on the nonsimply-connected Calabi-Yau manifold $X = Y/\mathbb{Z}_5$, where we quotient the quintic by the freely-acting $\mathbb{Z}_5$ symmetry (which exists for special choice of defining polynomial) given by the action

$$(x_1, x_2, x_3, x_4, x_5) \mapsto (\alpha x_1, \alpha^2 x_2, \alpha^3 x_3, \alpha^4 x_4, x_5), \qquad \alpha^5 = 1 \tag{3.44}$$

on the homogeneous coordinates of $\mathbb{P}^4$.

The line bundles on the quintic are entirely specified by their degree, and we denote them by $\mathcal{O}(n)$. On X, this is no longer true. The divisors $D_i = \{x_i = 0\}$ and $D_j = \{x_j = 0\}$ are no longer linearly-equivalent. Rather, there are flat, but nontrivial, line bundles

$$\mathcal{L}_{i-j} = \mathcal{O}(D_i - D_j) \tag{3.45}$$

Since $\mathbb{Z}_5$ is abelian, all its irreducible representations are one-dimensional, and the associated vector bundles, W_j are just the flat line bundles discussed above,

$$W_j = \mathcal{L}_j \tag{3.46}$$

More generally, the line bundles of degree n carry an extra label, $\mathcal{O}(n_\gamma)$, where $\gamma \in \mathbb{Z}/5\mathbb{Z}$. At degree-0, the flat line bundles, $\mathcal{L}_j$, form a group under tensor product, of which the trivial bundle, $\mathcal{O}$, is the identity element. When the degree $n \neq 0$, however, there is no natural choice of "origin" for the index γ.

The conformal field theory on X can be viewed as an orbifold of the conformal field theory on Y. As such, there is a $\mathbb{Z}_5$ quantum symmetry. This quantum symmetry acts on the D-branes (objects in the derived category) on X by tensoring with the flat line bundles, $\mathcal{L}_j$. The different choices of "origin" for the index γ are permuted by the action of the quantum symmetry group (the line bundles of degree n form a module for the quantum symmetry group).

On the quintic, Y, the monodromy about the large-radius limit is given by tensoring with $\mathcal{O}(1)$,

$$M_r(\mathcal{E}^\bullet) = (\mathcal{E} \otimes \mathcal{O}(1))^\bullet \equiv \mathcal{E}(1)^\bullet \tag{3.47}$$

On X, we need to *choose* a particular line bundle of degree-one, $\mathcal{O}(1_i)$,

$$M_r(\mathcal{E}^\bullet) = \mathcal{E}(1_i)^\bullet \tag{3.48}$$

Different choices of degree-one bundle are permuted by the quantum symmetry.

3.3.1 Monodromies about the conifold point

We have already written general formulæ for the monodromy about the conifold of wrapped D6-branes (3.42) and D0-branes (3.43). On the quintic , we have

$$\begin{aligned} M_c^Y(\mathcal{O}[0]) &\simeq \mathcal{O}[0] \\ M_c^Y(\mathcal{O}_p[0]) &\simeq \mathcal{I}_p[3]. \end{aligned} \tag{3.49}$$

On X,

$$\begin{aligned} M_c^X(\mathcal{L}_j[0]) &\simeq \mathcal{L}_j[0] \\ M_c^X(\mathcal{O}_p[0]) &\simeq \left({\bigoplus_j}' \mathcal{L}_j \oplus \mathcal{I}_p \right)[3] \end{aligned} \tag{3.50}$$

Now let's go further and consider other D-branes. We start with a D4-brane wrapped on hyperplane $D_i = \{x_i = 0\}$ in X. This D4-brane is represented by $\mathcal{O}_{D_i}[0]$. To calculate its monodromy, we look for a quasi-isomorphic complex $\mathcal{F}^\bullet$ with property (3.33).

Since the intersection of two hyperplanes, $\{x_j = 0\}$ and $\{x_k = 0\}$, in D_i is a point, p, the following complex is exact:

$$\begin{aligned} 0 \to \overset{-1}{\mathcal{O}_{D_i}} \xrightarrow{f} \mathcal{O}_{D_i}(2_{2j}) \oplus \mathcal{O}_{D_i}(2_{j+k}) \overset{0}{\oplus} \mathcal{O}_{D_i}(2_{j+k}) \oplus \mathcal{O}_{D_i}(2_{2k}) \\ \xrightarrow{g} \mathcal{O}_{D_i}(3_{2j+k}) \oplus \overset{1}{\mathcal{O}_{D_i}}(3_{j+2k}) \oplus \mathcal{O}_{D_i}(2_{j+k}) \xrightarrow{\rho} \overset{2}{S_p} \to 0. \end{aligned} \tag{3.51}$$

where

$$f = \begin{pmatrix} x_j^2 & x_j x_k & x_k x_j & x_k^2 \end{pmatrix}, \qquad g = \begin{pmatrix} -x_k & 0 & x_j & 0 \\ 0 & -x_k & 0 & x_j \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

S_p is the cokernel of g and ρ is the canonical projection. Actually, S_p is a skyscraper sheaf supported at p and its stalk at p is $\mathbb{C}^3$. Its module structure is not the standard one:

$$f(x) \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} f(p)\ a \\ f(p)\ b \\ f(p)\ c + \frac{\partial f}{\partial x_j}(p)\, a + \frac{\partial f}{\partial x_k}(p)\, b \end{pmatrix}. \tag{3.52}$$

Here, we have used the fact that in a small enough open neighbourhood of p, $\{x_i, x_j, x_k\}$ is a good coordinate system. With this representation of S_p, we can give an explicit formular for ρ:

$$\rho \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \alpha(p) \\ \beta(p) \\ \gamma(p) + \frac{\partial \alpha}{\partial x_j}(p) + \frac{\partial \beta}{\partial x_k}(p) \end{pmatrix} \tag{3.53}$$

Let $\mathcal{F}^\bullet$ be the complex obtained by replacing $\mathcal{O}_{D_i}$ at grade -1 of complex (3.51) with 0. $\mathcal{F}^\bullet$ is quasi-isomorphic to $\mathcal{O}_{D_i}[0]$. Furthermore, $\mathcal{F}^\bullet$ satisfies (3.33). To see that, one pulls back $\mathcal{F}^\bullet_{-l} = \mathcal{L}_l^* \otimes \mathcal{F}^\bullet$ to the quintic, the universal covering space of X. Then, $\pi^*\mathcal{F}^\bullet_{-l} = \pi^*\mathcal{F}^\bullet$, where $\pi : Y \to X$ is the canonical projection. Also, $\pi^*\mathcal{F}^\bullet$ has the property that each entry's higher cohomology vanishes. Since π^* is injective on cohomology, we conclude that $\mathcal{F}^\bullet_{-l}$ has the same property.

So, all we have to do is to calculate the cohomology of $\Gamma(\mathcal{F}^\bullet_{-l})$ and apply (3.37). The cohomology of $\Gamma(\mathcal{F}^\bullet_{-l})$ is given by the sheaf cohomology of $\mathcal{L}_l^* \otimes \mathcal{O}_{D_i}$ as before. Therfore, we have:

$$
\begin{aligned}
M_c^X(\mathcal{O}_{D_i}[0]) = 0 \to{}& \overset{-3}{\mathcal{O}} \xrightarrow{-f\circ r_{D_i}} \mathcal{O}_{D_i}(2_{2j}) \oplus \mathcal{O}_{D_i}(2_{j+k}) \overset{-2}{\oplus} \mathcal{O}_{D_i}(2_{j+k}) \oplus \mathcal{O}_{D_i}(2_{2k}) \\
& \xrightarrow{\binom{0}{g}} \bigoplus_l{}' \mathcal{L}_l \oplus \mathcal{O}_{D_i}(3_{2j+k}) \overset{-1}{\oplus} \mathcal{O}_{D_i}(3_{j+2k}) \oplus \mathcal{O}_{D_i}(2_{j+k}) \xrightarrow{(-\sigma\ \ \rho)} \overset{0}{S_p} \to 0 \\
& \simeq 0 \to \overset{-3}{\mathcal{O}} \xrightarrow{-f\circ r_{D_i}} \mathcal{O}_{D_i}(2_{2j}) \oplus \mathcal{O}_{D_i}(2_{j+k}) \overset{-2}{\oplus} \mathcal{O}_{D_i}(2_{j+k}) \oplus \mathcal{O}_{D_i}(2_{2k}) \\
& \xrightarrow{\binom{0}{g}} \mathrm{Ker}\, \overset{-1}{\begin{pmatrix} -\sigma & \rho \end{pmatrix}} \to 0.
\end{aligned}
\tag{3.54}
$$

where r_{D_i} is the restriction to D_i. Map $\sigma : \bigoplus_l{}' W_l \to S_p$ is defined by

$$
\sigma(f_l) = \tilde{ev}\Big(\sum_l{}' f_l \otimes s_{-l}\Big) \tag{3.55}
$$

where s_{-l} is a global section of $\mathcal{L}_l^* \otimes S_p$ which is not an image of $1 \otimes \rho$[h].

[h]They are unique upto image of $1 \otimes \rho$

We repeat the similar computation in the quintic. As before, a D4-brane wrapped on hyperplane D is represented by $\mathcal{O}_D[0]$. Its monodromy is:

$$
\begin{aligned}
M_c^Y(\mathcal{O}_D[0]) = 0 \rightarrow & \overset{-3}{\mathcal{O}} \xrightarrow{-f\circ\, r_D} \overset{-2}{\mathcal{O}_D(2)^{\oplus 4}} \\
& \xrightarrow{\binom{0}{g}} \overset{-1}{\mathcal{O}^{\oplus 4} \oplus \mathcal{O}_D(3)^{\oplus 2} \oplus \mathcal{O}_D(2)} \xrightarrow{(-\sigma\ \rho)} \overset{0}{S_{\{p_i\}}} \rightarrow 0 \\
& \simeq 0 \rightarrow \overset{-3}{\mathcal{O}} \xrightarrow{-f\circ\, r_D} \overset{-2}{\mathcal{O}_D(2)^{\oplus 4}} \xrightarrow{\binom{0}{g}} \overset{-1}{\operatorname{Ker}\begin{pmatrix}-\sigma & \rho\end{pmatrix}} \rightarrow 0.
\end{aligned} \tag{3.56}
$$

In this case, $S_{\{p_i\}}$ is supported at 5 points, p_i. Map $\sigma : \mathcal{O}^{\oplus 4} \rightarrow S_{\{p_i\}}$ is similarly defined as before with four global sections $\{s_{-l}\}$ of $S_{\{p_i\}}$ which are not images of ρ.

The next example we consider is a D2-brane. On X, the intersection of two hyperplanes is a genus 2 curve,

$$
C_{ij} = \{x_i = x_j = 0\}. \tag{3.57}
$$

A D2-brane wrapped on C_{ij} is represented by the sheaf $\mathcal{O}_{C_{ij}}[0]$, which has the following quasi-isomorphism:

$$
\begin{array}{cccccccc}
\mathcal{O}_{C_{ij}}[0]: & 0 & \longrightarrow & \overset{0}{\mathcal{O}_{C_{ij}}} & \longrightarrow & 0 & & \\
& & & \Big\downarrow{\scriptstyle x_k^3} & & & & \\
\mathcal{F}^\bullet \ : & 0 & \longrightarrow & \overset{0}{\mathcal{O}_{C_{ij}}(3_{3k})} & \xrightarrow{\rho} & \overset{1}{S_p} & \longrightarrow & 0.
\end{array} \tag{3.58}
$$

As in the D4-brane case, S_p is a skyscraper sheaf supported at point p and its stalk at p is $\mathbb{C}^3$. Its module structure is given by

$$f(x) \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} f(p)\, a \\ f(p)\, b + \frac{\partial f}{\partial x_k}(p)\, a \\ f(p)\, c + 2\frac{\partial f}{\partial x_k}(p)\, b + \frac{\partial^2 f}{\partial x_k^2}(p)\, a \end{pmatrix}. \tag{3.59}$$

With this representation, ρ is given by

$$\rho(\alpha) = \begin{pmatrix} \alpha(p) \\ \frac{\partial \alpha}{\partial x_k}(p) \\ \frac{\partial^2 \alpha}{\partial x_k^2}(p) \end{pmatrix} \tag{3.60}$$

Following the same procedure, we obtain:

$$\begin{aligned} M_c^X(\mathcal{O}_{C_{ij}}[0]) &= 0 \to \overset{-3}{\mathcal{O}} \xrightarrow{\begin{pmatrix} 0 \\ 0 \\ -x_k^3 \circ r_{C_{ij}} \end{pmatrix}} \overset{-2}{\mathcal{O}^{\oplus 2} \oplus \bigoplus_l{}' \mathcal{L}_l \oplus \mathcal{O}_{C_{ij}}(3_{3k})} \xrightarrow{(-\sigma \;\; -\tau \;\; \rho)} \overset{-1}{S_p} \to 0 \\ &\simeq 0 \to \overset{-3}{\mathcal{O}} \xrightarrow{\begin{pmatrix} 0 \\ 0 \\ -x_k^3 \circ r_{C_{ij}} \end{pmatrix}} \overset{-2}{\mathrm{Ker}\,(\,-\sigma \;\; -\tau \;\; \rho\,)} \to 0 \end{aligned} \tag{3.61}$$

Map $\sigma : \mathcal{O}^{\oplus 2} \to S_p$ and $\tau : \bigoplus_l{}' \mathcal{L}_l \to S_p$ are defined by

$$\begin{aligned} \sigma(f_1, f_2) &= \tilde{ev}(f_1 \otimes s_1 + f_2 \otimes s_2) && (3.62) \\ \tau(g_l) &= \tilde{ev}({\sum_l}' g_l \otimes t_{-l}) && (3.63) \end{aligned}$$

where s_i and t_{-l} are representatives of the cohomologies, $\mathrm{H}^1(\Gamma(\mathcal{F}^\bullet))$ and $\mathrm{H}^1(\Gamma(\mathcal{F}^\bullet_{-l}))$, respectively.

The computation for D2-branes in the quintic is almost identical. On Y, the intersection C of two hyperplanes is a genus 6 curve. Under the conifold monodromy, a D2-brane wrapped on C becomes

$$\begin{aligned} M_c^Y(\mathcal{O}_C[0]) &= 0 \to \overset{-3}{\mathcal{O}} \xrightarrow{\begin{pmatrix} 0 \\ 0 \\ -x_k^3 \circ r_C \end{pmatrix}} \overset{-2}{\mathcal{O}^{\oplus 6} \oplus \mathcal{O}_C(3)} \xrightarrow{(\, -\sigma \;\; \rho\,)} \overset{-1}{S_{\{p_i\}}} \to 0 \\ &\simeq 0 \to \overset{-3}{\mathcal{O}} \xrightarrow{\begin{pmatrix} 0 \\ 0 \\ -x_k^3 \circ r_C \end{pmatrix}} \overset{-2}{\mathrm{Ker}\,(\, -\sigma \;\; \rho\,)} \to 0 \end{aligned} \tag{3.64}$$

The map $\sigma : \mathcal{O}^{\oplus 6} \to S_{\{p_i\}}$ is similarly defined as in the previous case with 6 representatives $\{s_i\}$ of cohomology $\mathrm{H}^1(\Gamma(\mathcal{F}^\bullet))$.

For the last example, we consider a bound state of a D6-brane and an anti-D4-brane represented by

$$\mathcal{O}(-1_{-i})[0] \simeq 0 \to \overset{0}{\mathcal{O}} \xrightarrow{r_{D_i}} \overset{1}{\mathcal{O}_{D_i}} \to 0 \tag{3.65}$$

in X. The following is a quasi-isomorphic complex $\mathcal{F}^\bullet$ with property (3.33):

$$\begin{aligned} 0 \to \overset{0}{\bigoplus_{j,k} \mathcal{O}(1_{-i+j+k})} &\xrightarrow{f} \overset{1}{\bigoplus_{j,k} \mathcal{O}(2_{1-i-j+k}) \oplus \bigoplus_{j} \mathcal{O}(1_{1-i-j})} \\ &\xrightarrow{g} \overset{2}{\bigoplus_{j} \mathcal{O}(3_{1-i+j}) \oplus \mathcal{O}(2_{1-i})} \xrightarrow{\rho} \overset{3}{S_p} \to 0 \end{aligned} \tag{3.66}$$

where $j, k, \ldots$ run from 1 to 3 and, f and g are defined by

$$f \;\; : \;\; \{\alpha_{jk}\} \mapsto \left(\{\epsilon^{jlm} x_l \alpha_{mk}\}, \;\; \{\epsilon^{jkl}\alpha_{kl}\}\right) \tag{3.67}$$

$$g \;\; : \;\; \left(\{\beta^j_k\}, \;\; \{\beta^j\}\right) \mapsto \left(\{x_k \beta^k_j\}, \;\; x_j \beta^j - \beta^j_j\right). \tag{3.68}$$

As in the previous examples, S_p is a skyscraper sheaf supported at point p , the intersection of three hyperplanes $\{x_j = 0\}$, $j = 1, 2, 3$. Its stalk at p is $\mathbb{C}^4$

and the module structure is

$$f(x)\cdot\Big(\{a_j\},\quad b\Big) = \Big(\{f(p)\,a_j\},\quad f(p)\,b + \tfrac{\partial f}{\partial x_j}(p)\,a_j\Big)\,. \tag{3.69}$$

Again, $\{x_1, x_2, x_3\}$ is a good coordinate system around p. Then ρ is given by

$$\rho\left(\{\gamma_j\},\gamma\right) = \left(\{\gamma_j(p)\},\gamma(p) + \frac{\partial\gamma_j}{\partial x_j}(p)\right) \tag{3.70}$$

Now, the monodromy of the bound state is

$$\begin{aligned}
M_c^X(\mathcal{O}(-1_{-i})[0]) &= 0 \to \overset{-2}{\bigoplus_{j,k}\mathcal{O}(1_{-i+j+k})} \xrightarrow{f} \overset{-1}{\bigoplus_{j,k}\mathcal{O}(2_{1-i-j+k}) \oplus \bigoplus_{j}\mathcal{O}(1_{1-i-j})} \\
&\xrightarrow{\binom{0}{g}} \overset{0}{\bigoplus_{l}\mathcal{L}_l \oplus \bigoplus_{j}\mathcal{O}(3_{1-i+j}) \oplus \mathcal{O}(2_{1-i})} \xrightarrow{(\sigma\ \rho)} \overset{1}{S_p} \to 0 \\
&\simeq 0 \to \overset{-2}{\bigoplus_{j,k}\mathcal{O}(1_{-i+j+k})} \xrightarrow{f} \overset{-1}{\bigoplus_{j,k}\mathcal{O}(2_{1-i-j+k}) \oplus \bigoplus_{j}\mathcal{O}(1_{1-i-j})} \\
&\xrightarrow{\binom{0}{g}} \overset{0}{\mathrm{Ker}\,(\sigma\ \rho)} \to 0
\end{aligned} \tag{3.71}$$

where map $\sigma : \bigoplus_l \mathcal{L}_l \to S_p$ is given by

$$\sigma(f_l) = \tilde{ev}\Big(\sum_l f_l \otimes s_{-l}\Big) \tag{3.72}$$

and s_{-l} is a representative of $\mathrm{H}^3(\Gamma(\mathcal{F}^{\bullet}_{-l}))$.

The similar computation in quintic shows

$$\begin{aligned}
M_c^Y(\mathcal{O}(-1)[0]) &= 0 \to \overset{-2}{\mathcal{O}(1)^{\oplus 9}} \xrightarrow{f} \overset{-1}{\mathcal{O}(2)^{\oplus 9} \oplus \mathcal{O}(1)^{\oplus 3}} \\
&\xrightarrow{\binom{0}{g}} \overset{0}{\mathcal{O}^{\oplus 5} \oplus \mathcal{O}(3)^{\oplus 3} \oplus \mathcal{O}(2)} \xrightarrow{(\sigma\ \rho)} \overset{1}{S_{\{p_i\}}} \to 0 \\
&\simeq \overset{-2}{\mathcal{O}(1)^{\oplus 9}} \xrightarrow{f} \overset{-1}{\mathcal{O}(2)^{\oplus 9} \oplus \mathcal{O}(1)^{\oplus 3}} \xrightarrow{\binom{0}{g}} \overset{0}{\mathrm{Ker}\,(\sigma\ \rho)} \to 0
\end{aligned} \tag{3.73}$$

where map σ is defined similarly with five representatives of $\mathrm{H}^3(\Gamma(\mathcal{F}^\bullet))$.

3.3.2 Monodromies about the Landau-Ginsburg point

The remaining distinguished point in the Kähler moduli space of the quintic or its orbifold is the Landau-Ginsburg point. The monodromy about that point must satisfy

$$M_{LG} = M_c \circ M_r \tag{3.74}$$

So, on the quintic, the corresponding kernel is

$$K^{Y\bullet}_{LG} = 0 \to \overset{-3}{\mathcal{O} \boxtimes \mathcal{O}(1)} \xrightarrow{\tilde{r}} \overset{-2}{\mathcal{O}_\Delta(1)} \to 0 \tag{3.75}$$

Let us compute the orbit of the D6-brane, $\mathcal{O}[0]$ under the Landau-Ginsburg monodromy. Let $V = \mathrm{H}^0(Y, \mathcal{O}(1))$.

$$M^Y_{LG}(\mathcal{O}[0]) \simeq 0 \to \overset{-3}{\mathcal{O} \otimes V} \xrightarrow{ev} \overset{-2}{\mathcal{O}(1)} \to 0 \tag{3.76a}$$

Acting again, we get

$$\begin{aligned}(M^Y_{LG})^2(\mathcal{O}[0]) \simeq 0 &\to \overset{-6}{\mathcal{O} \otimes V \otimes V} \xrightarrow{\left(\begin{smallmatrix} sym \\ ev \end{smallmatrix}\right)} \overset{-5}{\mathcal{O} \otimes \mathrm{H}^0(Y, \mathcal{O}(2)) \oplus \mathcal{O}(1) \otimes V} \\ &\xrightarrow{\left(\begin{smallmatrix} -ev & ev \end{smallmatrix}\right)} \overset{-4}{\mathcal{O}(2)} \to 0 \\ &\simeq 0 \to \overset{-6}{\mathcal{O} \otimes \wedge^2 V} \xrightarrow{ev} \overset{-5}{\mathcal{O}(1) \otimes V} \xrightarrow{ev} \overset{-4}{\mathcal{O}(2)} \to 0\end{aligned} \tag{3.76b}$$

In similar fashion, we find

$$(M_{LG}^Y)^3(\mathcal{O}[0]) \simeq 0 \to \overset{-9}{\mathcal{O} \otimes \wedge^3 V} \to \overset{-8}{\mathcal{O}(1) \otimes \wedge^2 V} \to \overset{-7}{\mathcal{O}(2) \otimes V} \to \overset{-6}{\mathcal{O}(3)} \to 0 \quad (3.76c)$$

$$(M_{LG}^Y)^4(\mathcal{O}[0]) \simeq 0 \to \overset{-12}{\mathcal{O} \otimes \wedge^4 V} \to \overset{-11}{\mathcal{O}(1) \otimes \wedge^3 V} \to \overset{-10}{\mathcal{O}(2) \otimes \wedge^2 V}$$

$$\to \overset{-9}{\mathcal{O}(3) \otimes V} \to \overset{-8}{\mathcal{O}(4)} \to 0 \quad (3.76d)$$

$$(M_{LG}^Y)^5(\mathcal{O}[0]) \simeq 0 \to \overset{-15}{\mathcal{O} \otimes \wedge^5 V} \to \overset{-14}{\mathcal{O}(1) \otimes \wedge^4 V} \to \overset{-13}{\mathcal{O}(2) \otimes \wedge^3 V}$$

$$\to \overset{-12}{\mathcal{O}(3) \otimes \wedge^2 V \oplus \mathcal{O}} \to \overset{-11}{\mathcal{O}(4) \otimes V} \to \overset{-10}{\mathcal{O}(5)} \to 0 \quad (3.76e)$$

The apparent "extra" factor of $\mathcal{O}[12]$ in (3.76e) deserves explanation. The kernel of the map

$$\overset{-12}{\mathcal{O} \otimes \mathrm{H}^0(Y, \mathcal{O}(4)) \otimes V} \to \overset{-11}{\mathcal{O} \otimes \mathrm{H}^0(Y, \mathcal{O}(5))}$$

contains not just the usual antisymmetrized piece, but an additional piece from the derivatives of the quintic defining equation. This yields the "extra" factor of $\mathcal{O}$ in (3.76e).

The complexes (3.76) should be familiar to the cognoscenti. The bundle $T^*_{\mathbb{P}^4}(1)$ fits into the short exact sequence

$$0 \to T^*_{\mathbb{P}^4}(1) \to \mathcal{O} \otimes V \to \mathcal{O}(1) \to 0 \quad (3.77)$$

So (3.76a) is quasi-isomorphic to

$$M_{LG}^Y(\mathcal{O}[0]) \simeq T^*_{\mathbb{P}^4}(1)[3] \quad (3.78a)$$

And, similarly, the other complexes in (3.76) are resolutions of

$$(M_{LG}^{Y})^2(\mathcal{O}[0]) \simeq (\wedge^2 T^*_{\mathbb{P}^4})(2)[6] \tag{3.78b}$$

$$(M_{LG}^{Y})^3(\mathcal{O}[0]) \simeq (\wedge^3 T^*_{\mathbb{P}^4})(3)[9] \tag{3.78c}$$

$$(M_{LG}^{Y})^4(\mathcal{O}[0]) \simeq (\wedge^4 T^*_{\mathbb{P}^4})(4)[12] \simeq \mathcal{O}(-1)[12] \tag{3.78d}$$

$$(M_{LG}^{Y})^5(\mathcal{O}[0]) \simeq \mathcal{O}[12] \tag{3.78e}$$

While the shifts in grade may be unfamiliar, these are exactly the bundles conjectured in [42] (see also [20]) to be the "fractional branes" at the LG point of the quintic.

Completely analogous results holds for $X = Y/\mathbb{Z}_5$. The kernel corresponding to circling the Landau-Ginsburg point is

$$K_{LG}^{X\bullet} = 0 \to \overset{-3}{\bigoplus_{j=0}^{4} \mathcal{L}_j \boxtimes \mathcal{O}(1_{i-j})} \xrightarrow{\tilde{r}} \overset{-2}{\mathcal{O}_\Delta(1_i)} \to 0 \tag{3.79}$$

(Recall that the large-radius monodromy involved a choice of degree-one line bundle $\mathcal{O}(1_i)$.) There's a rank-4 bundle, F defined by the short exact sequence

$$0 \to F \to \bigoplus_{j=0}^{4} \mathcal{O}(-1_j) \to \mathcal{O} \to 0 \tag{3.80}$$

and the Landau-Ginsburg monodromies are

$$\begin{aligned}
M_{LG}^{X}(\mathcal{O}[0]) &\simeq F(1_i)[3] \\
(M_{LG}^{X})^2(\mathcal{O}[0]) &\simeq (\wedge^2 F)(2_{2i})[6] \\
(M_{LG}^{X})^3(\mathcal{O}[0]) &\simeq (\wedge^3 F)(3_{3i})[9] \\
(M_{LG}^{X})^4(\mathcal{O}[0]) &\simeq (\wedge^4 F)(4_{4i})[12] \simeq \mathcal{O}(-1_{4i})[12] \\
(M_{LG}^{X})^5(\mathcal{O}[0]) &\simeq \mathcal{O}[12]
\end{aligned} \tag{3.81}$$

Again, we find that M_{LG}^5 is the shift by 12 functor. Note that the dependence on the particular choice of degree-one bundle for the large-radius monodromy drops out when one takes the 5^{th} power of M_{LG}.

3.3.3 A new category

The Landau-Ginsburg monodromy *should* satisfy $M_{LG}^5 = 1$. Instead, we have found $M_{LG}^5 = [12]$, the shift-by-twelve functor. However shifting the grade by 6 is supposed to be a complete physical equivalence in the topological theory – it is just spectral flow in the open-string channel[i] – so perhaps we should be satisfied with this result.

But we cannot *really* be satisfied if we wish to cling strictly to the derived category.

$$\mathrm{Hom}_{D(X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \neq \mathrm{Hom}_{D(X)}(\mathcal{E}^\bullet[12], \mathcal{F}^\bullet)$$

for general $\mathcal{F}^\bullet$. So there is no obvious sense in which we should consider $\mathcal{E}^\bullet$ and $\mathcal{E}^\bullet[12]$ to be isomorphic.

However, there is a relatively simple modification of the derived category in which these are isomorphic. Let us define a new category, $\widetilde{D}(X)$, whose objects are again bounded complexes of coherent sheaves on a Calabi-

[i]Spectral flow by 3 takes you from the NS back to the NS sector, but with the opposite value of $(-1)^F$. That is, it turns branes into anti-branes. Spectral flow by 6 takes you back to the sector you started in, with the *same* value of $(-1)^F$.

Yau 3-fold, X, but whose morphisms are

$$\mathrm{Hom}_{\widetilde{D}(X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \bigoplus_{n=-\infty}^{\infty} \mathrm{Hom}_{D(X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet[6n]) \tag{3.82}$$

Since we are dealing with bounded complexes, the sum on n receives only a finite number of nonzero contributions.

These morphisms compose in the obvious way, given the isomorphism

$$\mathrm{Hom}_{D(X)}(\mathcal{E}^\bullet[k], \mathcal{F}^\bullet[k]) \simeq \mathrm{Hom}_{D(X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$$

Namely, if $F \in \mathrm{Hom}_{D(X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet[6n_1])$ and $G \in \mathrm{Hom}_{D(X)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet[6n_2])$, then $G \circ F$ is the obvious element of $\mathrm{Hom}_{D(X)}(\mathcal{E}^\bullet, \mathcal{G}^\bullet[6(n_1+n_2)])$.

We propose $\widetilde{D}(X)$ as the correct category of B-type topological open strings. This category takes account of the fact that shifting the grade by 6 should be a complete physical equivalence of the open string theory, whereas the original bounded derived category did not.

By *construction*, now, $\mathcal{E}^\bullet$ and $\mathcal{E}^\bullet[12]$ are isomorphic in the category $\widetilde{D}(Y)$ and hence represent the same D-brane. Which is to say that, in this new category, $M_{LG}^5 \simeq \mathbb{1}$.

None of our calculations in this paper are modified by this new proposal, though the formula (3.21) for the intersection-pairing can now be "simplified"

$$\begin{aligned}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) &= \sum_{i=-\infty}^{\infty} (-1)^i \dim \mathrm{Hom}_{D(X)}(\mathcal{F}^\bullet, \mathcal{E}[i]^\bullet) \\ &= \sum_{i=-2}^{+3} (-1)^i \dim \mathrm{Hom}_{\widetilde{D}(X)}(\mathcal{F}^\bullet, \mathcal{E}[i]^\bullet)\end{aligned} \tag{3.83}$$

This new category also explains a long-standing puzzle [23] about the correspondence between topological open strings and physical open strings. As you vary the Kähler moduli the relative grading between two D-branes can shift. Douglas required the grading to be $\mathbb{R}$-valued, so that, at a point where the branes are mutually-BPS, it is $\mathbb{Z}$-valued, in accordance with the derived category. This poses a puzzle because – when one wishes to untwist and recover the physical open strings – unitarity requires the charge of a chiral primary to lie in the range $0 \leq q \leq 3$.

In the original derived category, $\mathcal{E}^\bullet$ and $\mathcal{E}^\bullet[6]$ are not quasi-isomorphic, and hence represent *distinct* topological D-branes. But in $\widetilde{D}(X)$, they *are* isomorphic, and hence represent the *same* topological D-brane[j]. Similarly, the charge of open string states, rather than being $\mathbb{R}$-valued is only $\mathbb{R}/6$-valued. We can always choose the charge of an open string state to lie in the range $(-3, 3]$. In the physical theory, we interpret it as a chiral primary (if the charge is negative, we do a further spectral flow by 3 units, changing a brane to an anti-brane) whose charge is in the unitary range.

As explained in [23], if we look at the open string string theory stretched between a pair of branes $(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ as we vary the Kähler moduli, at some point we come to a place where the charge of one of the chiral primaries falls outside of the unitary range. At this point, that brane configuration *ceases to exist*[k].

[j] And $\mathcal{E}^\bullet[3] \simeq \mathcal{E}^\bullet[-3]$ is the corresponding anti-brane.

[k] It was already unstable (non-BPS); it simply continued to make sense as an unstable brane configuration up to this point.

In the present formulation, it is replaced by a *new* (anti-)brane configuration whose open string CFT is related by $N = 2$ spectral flow to the previous one.

There is still a 1-1 correspondence between D-branes and *isomorphism classes of objects* in $\widetilde{D}(X)$ and between chiral primaries and morphisms in $\widetilde{D}(X)$. There would be no such correspondence (as argued by Douglas [23]), if we used the category $D(X)$. The point is that there are *more* isomorphisms in $\widetilde{D}(X)$ and hence *fewer* isomorphism classes. Objects which represented distinct topological D-branes in $D(X)$ are isomorphic in $\widetilde{D}(X)$ and hence represent the same D-brane. The cost, however, is that the correspondence (the twist required to go from the topological theory back to the physical one) jumps discontinuously as we move in the moduli space.

Chapter 4

F-theory Compactification

In this chapter, we consider F-theory compactified on an elliptically fibered Calabi-Yau 4-fold X, and its moduli stabilization, closely following KKLT's arguments. We will review how complex structure moduli are stabilized by the flux and also consider non-perturbative effects that fix Kähler moduli. As pointed out in [18], these non-perturbative effects do not generally fix all Kähler moduli. But there are some cases where all Kähler moduli are frozen. Grassi classified all those cases for fano bases. In [18], a couple of toric bases are considered. Here, we consider a non-toric example and perform explicit analytic computations to show that the overall size of X is big enough for the supergravity approximation to be valid.

4.1 F-theory Compactification

Let's start by reviewing the various properties of the compactification. Since X is elliptically fibered, we have fibration π:

$$\begin{tikzcd} T \arrow[r] & X \arrow[d, "\pi"'] \\ & B \end{tikzcd} \tag{4.1}$$

where T is the fiber, a torus and B is the base. F-theory compactified on X is defined to be Type IIB String Theory on B with non-trivial τ background and 7-branes. The elliptic fibration encodes the background τ and the positions of the 7-branes. The modulus of the fiber will vary as we move along the base and it is identified with the background τ. The positions and types of the 7-branes are determined by the singular loci of τ and its monodromies around them.

Note the modulus of a torus is only determined up to $\mathrm{SL}(2,\mathbb{Z})$ actions. This $\mathrm{SL}(2,\mathbb{Z})$ ambiguity is identified with the $\mathrm{SL}(2,\mathbb{Z})$ self-duality of Type IIB String Theory. This implies that there is the same $\mathrm{SL}(2,\mathbb{Z})$ ambiguity[a] in H and F, the NS-NS and R-R 3 form field strengths. Mathematically, this can be formulated in the following way. Let B' be B minus the worldvolume of 7-branes. For each point $p \in B'$, one can consider the first cohomology group of the fiber, $\mathrm{H}^1(X_p,\mathbb{Z})$. The collection of these cohomology groups forms a fibration over B':

$$\begin{array}{ccc} \mathrm{H}^1(T,\mathbb{Z}) & \longrightarrow & E \\ & & \downarrow \\ & & B'. \end{array} \tag{4.3}$$

Let $E_{\mathbb{R}} = E \otimes \mathbb{R}$. $E_{\mathbb{R}}$ is a flat rank 2 vector bundle with structure group $\mathrm{SL}(2,\mathbb{Z})$. Then, τ is a section of $\mathbb{P}(E^*_{\mathbb{R}})$, the projectivization of the dual bundle

[a] Under a $\mathrm{SL}(2,\mathbb{Z})$ element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\tau \to \frac{a\tau+b}{c\tau+d}, \qquad \begin{pmatrix} H \\ F \end{pmatrix} \to \begin{pmatrix} d & -c \\ a & -b \end{pmatrix} \begin{pmatrix} H \\ F \end{pmatrix} \tag{4.2}$$

of $E_{\mathbb{R}}$ and H and F are components of a section of $\Omega^3 \otimes E_{\mathbb{R}}$. Actually, a more natural object describing H and F is the 4-form field strength $G_F^{(4)}$. It is a 4-form in X. To define this, we need to introduce a local basis $\{\theta_1(p), \theta_2(p)\}$ of $\mathrm{H}^1(X_p, \mathbb{Z})$ with property $\int_{X_p} \theta_1(p) \wedge \theta_2(p) = 1$ for every $p \in B'$. θ_1 and θ_2 can be considered as locally defined harmonic 1-forms in X. We also pull-back H and F to X and consider them as degree 3 local forms in X, too. Then, we define $G_F^{(4)}$ by

$$G_F^{(4)} = \frac{F}{2\pi\alpha'} \wedge \theta_1 + \frac{H}{2\pi\alpha'} \wedge \theta_2. \tag{4.4}$$

Note that the $\mathrm{SL}(2, \mathbb{Z})$ ambiguity in the definition of $\{\theta_1, \theta_2\}$ compensates the $\mathrm{SL}(2, \mathbb{Z})$ ambiguity in H and F to make $G_F^{(4)}$ a globally defined 4-form in X. Obviously, not every 4-form in X can be realized as $G_F^{(4)}$. Only 4-forms that have exactly one "leg" in the fiber direction are qualified. To describe this invariantly, one uses the fact that every elliptic fibration has a section $s : B \to X$. Then, $G_F^{(4)}$ must satisfy $\pi_* G_F^{(4)} = 0$ and $s^* G_F^{(4)} = 0$. As we will see later, various quantities in the F-theory compactification are more naturally expressed in term of $G_F^{(4)}$ rather than H and F.

If we compactify this theory further on a circle, it is conjectured to be dual to M-theory compactified on X. The M-theory 4 form field strength $G^{(4)}$ is identified with $G_F^{(4)}$ we just discussed. In the limit where the size of the fiber becomes zero, we will recover the original F-theory. This is a useful way of determining various properties of the F-theory compactification and we will use this approach.

From the M-theory side, it is easy to see there are 4 unbroken supercharges. The F-theory compactification should have the same number of supercharges, since it is a special limit of the M-theory compactification. Hence, we have $N = 1$ supersymmetry in 4 dimensions. Also, the spectrum of light particles can be easily investigated on the M-theory side. They come from the moduli of the Kähler structure and complex structure of X. Each Kähler structure modulus gives rise to a 3 dimensional linear multiplet which contains one scalar and one 3 dimensional vector[53]. The scalar comes from the volume of a divisor D in X while the vector is obtained by integrating the M-theory 3-form C along $*D$, the Poincare dual cycle of D. One can dualize the vector to get a dual scalar, then the linear multiplet becomes a conventional chiral multiplet. For each complex structure modulus of X, we will get a chiral multiplet. The complex scalar in the chiral multiplet comes from the 11 dimensional metric. In the F-theory limit, we set the size of the torus to zero. Hence, the corresponding Kähler structure modulus is not present in the F-theory compactification. We will get a 4 dimensional chiral superfield for each Kähler structure modulus of B. It contains two scalars, one from the volume of a divisor A in B and another from integrating the RR 4-form potential along A. We also have chiral superfields from the complex structure moduli of B and the moduli coming from changing the τ background and the positions of 7-branes. These moduli have a common M-theory origin—they all come from the complex structure moduli of X.

4.2 Moduli Stabilization

Now, we get back to the M-theory side and see how the moduli are stabilized. To stablize the complex structure moduli and some Kähler moduli, we turn on the M-theory 4-form flux, $G^{(4)}$ and include M2 branes in our background. Before analyzing their effects, we review a couple of constraints on them. First, there is a quantization condition on $G^{(4)}$:

$$\left[\frac{G^{(4)}}{2\pi}\right] - \frac{\omega_4}{2} \in \mathrm{H}^4(X, \mathbb{Z}) \tag{4.5}$$

where ω_4 is the fourth Stieffel-Whitney class. As we will see later, our example has even ω_4. In that case, $\left[\frac{G^{(4)}}{2\pi}\right]$ is simply an element in the integral cohomology. There is another constraint coming from the fact X is compact:

$$N_{\mathrm{M2}} = \frac{\chi(X)}{24} - \frac{1}{2}\int_X \frac{G^{(4)}}{2\pi} \wedge \frac{G^{(4)}}{2\pi} \tag{4.6}$$

where $\chi(X)$ is the Euler characteristic of X and N_{M2} is the net charge of M2 branes that fill the non-compact dimensions.

To preserve 3 dimensional Poincare symmetry, we assume all M2 branes fill the non-compact directions and use the following ansatz for $G^{(4)}$:

$$G^{(4)} = G^{(4)}_X + \mu \wedge \mathrm{d}f \tag{4.7}$$

where $G^{(4)}_X$ is a 4 form in X, μ is the volume form of the 3 dimensional spacetime, and f is a real valued function on X. In [36], using the 11 dimensional supergravity approximation, it is shown that the supersymmetry is not broken further by $G^{(4)}$ and M2 branes if the following conditions are met:

1. $G_X^{(4)}$ is a primitive closed (2,2) form in X.

2. There is no anti-M2 branes.

The first condition should be understood as a constraint on the complex structure and Kähler structure of X for a given $G_X^{(4)}$. In this level of approximation, we have no-scale supergravity as the low enery effective theory and the first condition can be imposed by introducing the superpotential[36]:

$$W = \int_X G^{(4)} \wedge \Omega_X + \int_X k_X \wedge k_X \wedge G^{(4)} \tag{4.8}$$

where Ω_X is the holomorphic 4-form and k_X is the Kähler form of Calabi-Yau, X. Generally, this superpotential stablizes all complex structure moduli and some Kähler moduli. Also, the metric on X will pick up a warp factor. $G_X^{(4)}$ and the positions of M2 branes uniquely determine this warp factor and f leaving the positions of M2 branes as moduli. These additional moduli fields can be avoided if we carefully choose $G_X^{(4)}$ so that there are no M2 branes allowed in our background via (4.6).

The F-theory side has a similar story. We can turn on H and F fluxes and include space-time filling D3 branes in our background. As before there are two constraints. One is the quantization condition and the other is the condition that the total D3 brane charge must be zero. As we discussed earlier, the natural object describing F and H is $G_F^{(4)}$ and it is identified with $G^{(4)}$ in the M-theory. Therefore, the quantization condition is simply

$$\left[\frac{G_F^{(4)}}{2\pi}\right] - \frac{\omega_4}{2} \in \mathrm{H}^4(X, \mathbb{Z}) \tag{4.9}$$

and the total D3 brane charge condition becomes

$$\begin{aligned} N_{\mathrm{D3}} &= \frac{\chi(X)}{24} - \frac{1}{2}\int_X \frac{G_F^{(4)}}{2\pi} \wedge \frac{G_F^{(4)}}{2\pi} \\ &= \frac{\chi(X)}{24} - \frac{1}{(2\pi)^4(\alpha')^2}\int_B H \wedge F \end{aligned} \tag{4.10}$$

where N_{D3} is the net charge of D3 branes that fill non-compact directions. The superpotential generated by introducing $G_F^{(4)}$ flux is similarly derived from the M-theory counter part:

$$W = \int_X G_F^{(4)} \wedge \Omega_X \tag{4.11}$$

since the second term of (4.8) is identically zero in this case. This superpotential generally stabilize all moduli that have their origin in the complex structure moduli of X.

In [39], it is argued that the Kähler structure moduli can be stabilized by non-perturbative corrections to the superpotential. The first non-perturbative correction we will consider here was discovered by Witten[53] and it is present in both M-theory and F-theory compactification. In the M-theory compactification, one can consider a Euclidean M5 brane wrapping on an irreducible hypersurface D of X. In a 3-dimensional observer's point of view, it is an instanton. Witten showed that this configuration's R-symmetry charge is 2 times $\chi(D, \mathcal{O}_D)$, the arithmetic genus of D. When the hypersuface's arithmetic genus is 1, there is the following correction to the superpotential:

$$W_{\mathrm{inst}} = T(z)\exp(-\rho_D) \tag{4.12}$$

where ρ_D is the chiral superfield whose real part is the volume of D and $T(z)$ is a complex structure dependent one-loop determinant. In F-theory, this correction arises when a Euclidean D3 brane wrapping on an irreducible hypersurface A of B and $\pi^{-1}(A)$ has arithmetic genus 1. For the M-theory compactification, there always exists a hypersurface satisfying the above condition. That is $s(B)$, the image of B under the section s. Note that the Kähler modulus associated to this divisor is the size of the fiber. However, this configuration does not survive when we take the F-theory limit. We do not include this correction in the F-theory compactification.

The other non-perturbative correction considered in [39] is the gaugino condesation on 7-branes present in the F-theory compactification. Suppose we choose $G_F^{(4)}$ carefully and the superpotential (4.8) fixes the complex structure of X such that there is a stack of N_c 7-branes wrapping on a hypersurface A of B. Further assume that there are no moduli in moving the positions of the 7-branes. Then, the low energy effective theory on the worldvolume of the 7-branes is a 4 dimensional $N = 1$ supersymmetric pure $\mathrm{SU}(N_c)$ Yang-Mills. The gauge coupling of this theory is

$$\frac{8\pi^2}{g_{\mathrm{YM}}} = V(A) \tag{4.13}$$

where $V(A)$ is the volume of A. The gaugino condensation on the 7-branes yields the following correction to the superpotential:

$$W_{\mathrm{gauge}} = T(z)\exp(-\frac{\rho_A}{N_c}). \tag{4.14}$$

where ρ_A is the Kähler moduli superfield whose real part is $V(A)$. Actually, this non-perturbative effect can be understood on the M-theory side[40], too.

Because of the non-abelian gauge group on 7-branes on the F-theory side, X is not smooth. The fibration over A is so singular that X itself becomes singular, too. To obtain a smooth Calabi-Yau, $\tilde{X}$, one resolves this singularity and gets exceptional divisors. These divisors have arithmetic genus 1 and M5 branes wrapping on one of these exceptional divisors give the same correction to the superpotential.

The non-perturbative effects we considered so far give rise to exponential superpotential to the chiral superfields corresponding to the Kähler moduli. The generation of such superpotentials will ruin the no-scale structure of the low energy effective theory. Previously, with the no-scale assumption, the supersymmetry is unbroken only if $G^{(4)}$ is a primitive (2,2) form and this condition is imposed by the superpotential (4.8). With the corrections we considered here, the supersymmetric solutions might have non-vanishing (4,0) part of $G^{(4)}$, generally leading to an anti-de Sitter vacuum. In [39], it is argued that, with careful choice of the flux, these corrections stabilize all Kähler moduli of our background. However, as pointed out in [18], such cases are rare. Actually, if B has only one Kähler modulus, it cannot be stabilized by this mechanism. Suppose the contrary. Then, there exists an irreducible hypersurface A in B such that $D = \pi^{-1}(A)$ has a positive aritmetic genus. In [22], it is shown that such A is non-nef. That means there exists a holomorphic curve C that has a negative intersection number with A. Since there is only one Kähler modulus, $[C] = p[A]^2$ and $k_B = r[A]$ for some $p, r \in \mathbb{R}$. Here, $[S]$ denotes the cohomology element that is Poincare dual to cycle S. The volumes of B, A, and C are positive:

$$\begin{aligned}
\mathrm{Vol}(B) &= \int_B k_B^3 = r^3 A^3 > 0 \\
\mathrm{Vol}(A) &= \int_A k_B^2 = r^2 A^3 > 0 \\
\mathrm{Vol}(C) &= \int_C k_B = rpA^3 > 0.
\end{aligned} \tag{4.15}$$

From the above, we conclude that $r > 0, p > 0$, and $A^3 > 0$. However, it implies $C \cdot A = pA^3 > 0$, a contradiction. Hence, for any B with $h^{(1,1)} = 1$, the Kähler modulus cannot be stabilized in this way. Acutally, Grassi[30] classified all fano B whose Kähler moduli can be stabilized by the non-perturbative effects we considered here. In her analysis, she assumed there is no flux. With $G_F^{(4)}$ turned on, we need to extend the argument a little bit. For M5(or D3) branes to be supersymmetric, they must wrap holomorphic cycles in X(or B) when there is no flux. Presumably, this condition can be derived from the supergravity equations of motion. In the presence of the flux, the supersymmetric cycle condition might change. However, since it is derived from the equations of motion and the equations of motion are insensitive to the quantization of the flux, the supersymmetric cycle condition must vary continuously with the flux. Hence, the topological condition that was derived with the assumption that there is no flux is still valid even with the flux turned on.

There is an additional condition that hypersurfaces should satisfy to generate the superpotential when the flux is turned on. That is $G^{(4)}$ restricted to a given hypersurface D must be trivial in cohomology. The worldvolume theory of M5 branes contains a self-dual three form field strength T. Its equation of motion reads:

$$\mathrm{d}T = G^{(4)}|_D - 2\pi\delta(\partial M) \tag{4.16}$$

where M is the worldvolume of M2 branes ending on D and ∂M is the union of all boundaries of such M2 branes. In our case, all M2 branes fill non-compact directions and generally, ∂M is empty. This additional condition on D is consistent with the fact that $k_X \wedge G^{(4)} = 0$ for unbroken supersymmetry. If $G^{(4)}|_D$ were non-trivial and D generated the superpotential, then the superpotential might lead to a vacuum where $G^{(4)}$ is no longer primitive. Hence, those hypersurfaces should not generate any superpotential. In the F-theory limit, for any divisor A of B, $G_F^{(4)}|_{\pi^{-1}(A)}$ is cohomologically trivial. This can be checked from the fact B does not have any holomorphic one form[b] and hence, any (2,1) form on B is primitive for any given Kähler structure. Therefore, we can take the result of Grassi without any modifications. In [18], the cases where B is toric are considered. Unfortunately, their examples have $h^{(1,1)} = 5$ and 3. Hence, it is extremely difficult to find a vacuum analytically and the authors of the paper relied on numerical analysis. Here, we consider a case where B is non-toric, but simple enough to have an analytic vacuum solution for Kähler modulus.

[b]If it had one, one could pull-back this to X to obtain holomorphic one form in X. However, X does not have one.

4.3 Weierstrass Model

In this section, we analyze the Weierstrass model for a given base B. We start from vector bundle V that is a direct sum of 3 line bundles, L_1, L_2, and L_3 over B. The projectivization $\mathbb{P}(V)$ of V is a $\mathbb{P}^2$ bundle. We will construct an elliptically fibered Calabi-Yau 4-fold X as a hypersurface in $\mathbb{P}(V)$. Before that, we would like to compute the cohomology and Chern class of the ambient space $\mathbb{P}(V)$. Consider the pull-back bundle $\pi^{-1}V$ over $\mathbb{P}(V)$:

$$\begin{array}{ccc} \pi^{-1}V & & V \\ \downarrow & & \downarrow \\ \mathbb{P}(V) & \xrightarrow{\pi} & B \end{array} \tag{4.17}$$

Over $\mathbb{P}(V)$, there is a short exact sequence of vector bundles:

$$0 \to T \to \pi^{-1}V \to Q \to 0. \tag{4.18}$$

where T is the tautological line bundle and Q is the quotient bundle. Note that any point in $\mathbb{P}(V)$ is a pair (p, l) where p is a point in B and l is a line in V_p. The tautological line bundle T is a sub-bundle of V whose fiber over (p, l) is the set of points on l. This is the exactly same situation when one blows up along B. There, V is the normal bundle of B, $\mathbb{P}(V)$ is the exceptional divisor and T is the normal bundle of $\mathbb{P}(V)$.

Let $\eta = -c_1(T)$. From the definition of the Chern class, $c(V)$, we have:

$$\mathrm{H}^\bullet(\mathbb{P}(V)) \simeq \mathrm{H}^\bullet(B)[\eta]/\{\eta^3 + c_1(V)\eta^2 + c_2(V)\eta + c_3(V) = 0\}. \tag{4.19}$$

Also, on each $p \in B$

$$\int_{\mathbb{P}(V)_p} \eta^2 = 1. \tag{4.20}$$

This is because $\mathbb{P}(V)_p \simeq \mathbb{P}^2$ and η restricted to $\mathbb{P}(V)_p$ is the hyperplane class of $\mathbb{P}^2$.

Now, having computed the cohomology of $\mathbb{P}(V)$, let's calculate its Chern class. There is another exact sequence of vector bundles over $\mathbb{P}(V)$:

$$0 \to T\mathbb{P}(V)_{\text{vert}} \to T\mathbb{P}(V) \to \pi^{-1}TB \to 0 \tag{4.21}$$

where $T\mathbb{P}(V)_{\text{vert}}$ is the collection of the vertical tangent vectors of $\mathbb{P}(V)$. Therefore, $c\left(\mathbb{P}(V)\right) = c\left(T\mathbb{P}(V)_{\text{vert}}\right) \wedge c(B)$[c]. Note that $T\mathbb{P}(V)_{\text{vert}} = Q \otimes T^*$. By tensoring (4.18) with T^*, we obtain an exact sequence $0 \to \mathbb{C} \to \pi^{-1}V \otimes T^* \to Q \otimes T^* \to 0$. Hence,

$$c\left(\mathbb{P}(V)\right) = c\left(\pi^{-1}V \otimes T^*\right) \wedge c(B). \tag{4.22}$$

To define X, we introduce the homogeneous coordinates, $Z_i, i = 1, 2, 3$ along the $\mathbb{P}^2$-fiber. It turns out that Z_i is a section of the line bundle, $\pi^{-1}L_i \otimes T^*$ over $\mathbb{P}(V)$. In the Weierstrass model, X is the set of points in $\mathbb{P}(V)$ that satisfy

$$Z_2^2 Z_3 = Z_1^3 + f Z_1 Z_3^2 + g Z_3^3 \tag{4.23}$$

where f and g are sections of some pull-back line bundles $\pi^{-1}F$ and $\pi^{-1}G$. The above equation makes sense only when $L_2^2 \otimes L_3 = L_1^3$, $F = L_1^{-1} \otimes L_2^2 \otimes L_3^{-1}$, and $G = L_2^2 \otimes L_3^{-1}$. Since we want X to be a Calabi-Yau, we set $c_1\left(\mathbb{P}(V)\right) =$

[c]From now on, we abuse the notation and implicitly embed $\mathrm{H}^\bullet(B)$ into $\mathrm{H}^\bullet(\mathbb{P}(V))$ via (4.19).

$2c_1(L_2) + c_1(L_3)$. From the first Chern class of the ambient space, $c_1(\mathbb{P}(V)) = c_1(L_1) + c_1(L_2) + c_1(L_3) + 3\eta + c_1(B)$, we have

$$\begin{aligned} c_1(F) &= 4c_1(B) \\ c_1(F) &= 6c_1(B) \\ c_1(L_1) &= c_1(L_3) + 2c_1(B) \\ c_1(L_2) &= c_1(L_3) + 3c_1(B). \end{aligned} \tag{4.24}$$

Note that $c_1(L_3)$ is not determined by this argument. This is because $\mathbb{P}(V)$ is independent of the choice of L_3. $\mathbb{P}(V \otimes L) = \mathbb{P}(V)$ for any line bundle L. However, the definitions of T and η depend on the choice of L_3. What is invariant is $T \otimes \pi^{-1}L_3^{-1}$ and $\eta + c_1(L_3)$. For simplicity, we set $L_3 = \mathbb{C}$.

With these assignments, the divisor class of X in $\mathbb{P}(V)$ is $3\eta + 6c_1(B)$. Since $c_1(V) = 5c_1(B)$ and $c_2(V) = 6c_1(B)^2$, it can be easily checked that $\eta^2 = -3c_1(B)\eta$ on X. Now, we can calculate the Chern class of X,

$$\begin{aligned} c_1(X) &= 0 \\ c_2(X) &= 11c_1(B)^2 + c_2(B) + 4c_1(B)\eta \\ c_3(X) &= -60c_1(B)^3 - c_1(B)c_2(B) + c_3(B) - 20c_1(B)^2\eta \\ c_4(X) &= (120c_1(B)^3 + 4c_1(B)c_2(B))\eta. \end{aligned} \tag{4.25}$$

We are interested in determining the non-perturbative corrections to the superpotential. To find which Kähler moduli are stabilized, we would like to compute the arithmetic genus of $D = \pi^{-1}(A)$ for a given irreducible hypersurface A in B. In [22], it is shown that

$$\chi(D, \mathcal{O}_D) = -\frac{1}{24}[D]^2 c_2(X) \tag{4.26}$$

for such a D. From the results of the above analysis,

$$\begin{aligned}\chi(D, \mathcal{O}_D) &= -\frac{1}{24}\int_X [D]^2 c_2(X) \\ &= -\frac{1}{24}\int_{\mathbb{P}(V)} [D]^2(11c_1(B)^2 + c_2(B) + 4c_1(B)\eta)(3\eta + 6c_1(B)) \\ &= -\frac{1}{2}\int_B [A]^2 c_1(B)\end{aligned} \tag{4.27}$$

Therfore, only irreducible hypersurfaces A that satisfy $\int_B [A]^2 c_1(B) = -2$ enter in the superpotential.

4.4 An Example

In this section, we apply the previous section's results to an example and show that all Kähler moduli are stabilized and the overall volume of B is large. Mori and Mukai classified all fano 3-folds with $b_2 \geq 2$[43, 44]. The base we consider here is Example 21 in their list with $b_2 = 2$. It is the blow-up of a quadric, Q, in $\mathbb{P}^4$ with center a twisted quartic, a smooth rational curve of degree 4 which spans $\mathbb{P}^4$ on it. Let's start with a twisted quartic curve C in $\mathbb{P}^4$. Consider a map from $\mathbb{P}^1 \to \mathbb{P}^4$:

$$[u, v] \mapsto [u^4, u^3v, u^2v^2, uv^3, v^4], \qquad \forall [u, v] \in \mathbb{P}^1. \tag{4.28}$$

The image of this map in $\mathbb{P}^4$ is C. Now, we choose a quadric Q in $\mathbb{P}^4$ containing C. The even degree cohomology of Q is generated by $[H]$, the hyperplane class. Its Chern class is $c(Q) = 1 + 3[H] + 4[H]^2 + 2[H]^3$. Also, one can easily check $H^3 = 2$ in Q. A generic hyperplane intersects with C at 4 points. Since C

is diffeomorphic to $\mathbb{P}^1$ and $\chi(C) = 2$, we conclude that $c(C) = 1 + \frac{1}{2}[H]$ and $c(N_{C|Q}) = 1 + \frac{5}{2}[H]$ where $N_{C|Q}$ is the normal bundle of C in Q.

The fano 3-fold B is the blow-up of Q along C. To compute its cohomology and Chern class, we consider the following:

$$\begin{array}{ccc} \sigma^{-1}N_{C|Q} & & N_{C|Q} \\ \downarrow & & \downarrow \\ E = \mathbb{P}(N_{C|Q}) & \xrightarrow{\sigma} & C \end{array} \tag{4.29}$$

where E is the exceptional divisor. Following exactly the same procedures as in the previous section, we obtain:

$$\mathrm{H}^\bullet(E) \simeq \mathrm{H}^\bullet(C)[\zeta]/\{\zeta^2 + \frac{5}{2}H\zeta = 0\} \tag{4.30}$$

$$\int_E \zeta \wedge [H] = 4. \tag{4.31}$$

Here, $\zeta = -c_1(T)$ and T is the tautological sub-bundle of $\sigma^{-1}N_{C|Q}$. As we discussed in the previous section, the normal bundle $N_{E|B}$ of E in B is T. Therefore, $[E]|_E = -\zeta$. $[H]$ and $[E]$ generate the even degree cohomology of B. It is easy to check that in B

$$H^3 = 2, \quad H^2E = 0, \quad HE^2 = -4, \quad E^3 = -10. \tag{4.32}$$

The Chern class of B is also computed[35]: $c(B) = 1 + (3[H] - [E]) + (6[H]^2 - 3[H][E]) + 2[H]^3$.

To show that the Calabi-Yau 4-fold X constructed from B has even ω_4, we first note that $\omega_4 = c_2(X) \mod 2$. From (4.25), one concludes that ω_4 is even if and only if $c_1(B)^2 + c_2(B)$ is even. $c_1(B)^2 + c_2(B) = 15H^2 - 9HE + E^2$

and from (4.32), it has even intersection numbers with any integral cohomology elements. Therefore, it is even and so is ω_4.

Having computed all these topological data of B, now we can compute the superpotential. Let A be any divisor in B with $[A] = n[H] + m[E]$ and $D = \pi^{-1}(A)$. Then,

$$\begin{aligned} \chi(D, \mathcal{O}_D) &= -\frac{1}{2}\int_B [A]^2 c_1(B) \\ &= (2n-m)^2 - 7n^2. \end{aligned} \tag{4.33}$$

Among divisors with $(2n-m)^2 - 7n^2 = 1$, only $3H - 2E$ and E are irreducible hypersurfaces as found by Grassi.

Before writing the superpotential for our model, we would like to point out that all the topological data we have obtained so far is invariant under the following map:

$$H \mapsto 2H - E \qquad E \mapsto 3H - 2E. \tag{4.34}$$

Under this map, $3H - E$ and E are exchanged. Therefore, we conclude that $3H - E$ and E have the same topological data, even including their normal bundles. Since the superpotential only depends on the topological data of B, we conclude that it is also invariant under this map.

Following [18], we write the Kähler class k_B of B as

$$k_B = t_1[A_1] + t_2[A_2]. \tag{4.35}$$

where $A_1 = H$ and $A_2 = 2H - E$. We choose $[H]$ and $[2H - E]$ as our basis because of the above symmetry. Under (4.34), they are exchanged. As argued in [18], t_i's are not good coordinate in the F-theory compactification. The volumes $V_i = \frac{1}{2}\sum_{j,k} A_iA_jA_kt_jt_k$ of our basis divisors A_i are better coordinates:

$$\begin{aligned} V_1 &= t_1^2 + 4t_1t_2 + 2t_2^2 \\ V_2 &= 2t_1^2 + 4t_1t_2 + t_2^2. \end{aligned} \tag{4.36}$$

There are corresponding chiral superfields, ρ_1 and ρ_2. Their scalar components, which we denote with the same symbols as their superfields by abuse of notation, can be written as

$$\rho_i = V_i - i\int_{A_i} C^{(4)} \tag{4.37}$$

where $C^{(4)}$ is the RR 4-form potential in Type IIB String Theory.

Now, we write the superpotential:

$$W = W_0 + T_{(-1,2)}\exp[\rho_1 - 2\rho_2] + T_{(2,-1)}\exp[-2\rho_1 + \rho_2]. \tag{4.38}$$

Here, we assume that the complex moduli are already stabilized at a higher energy scale and treat $W_0, T_{(-1,2)}$ and $T_{(2,-1)}$ as constants. We further assume that the map (4.34) can be extended to the entire cohomology and $G_F^{(4)}$ is invariant under it. With these assumptions, we conclude that $T_{(-1,2)} = T_{(2,-1)}$ and drop the subscripts from now on.

A supersymmetric vacuum is a solution to

$$0 = D_iW = \partial_iW + (\partial_iK)\,W \tag{4.39}$$

where K the Kähler potential. In the supergravity approximation, $K = -2\log(V)$ where V is the volume of X. In terms of the t_i's,

$$V = \frac{1}{6}\sum_{i,j,k} A_iA_jA_kt_it_jt_k = \frac{1}{3}\left(t_1^3 + 6t_1^2t_2 + 6t_1t_2^2 + t_2^3\right). \tag{4.40}$$

With $V_i = \frac{\partial V}{\partial t_i}$ and $V_i t_i = 3V$,

$$\begin{aligned}\frac{\partial V}{\partial V_i} &= \left(\frac{\partial t_j}{\partial V_i}\right) V_j \\ &= \frac{\partial}{\partial V_i}(t_j V_j) - t_i \\ &= 3\frac{\partial V}{\partial V_i} - t_i \\ &= \frac{1}{2} t_i\end{aligned} \tag{4.41}$$

Hence,

$$\begin{aligned}\frac{\partial K}{\partial \rho_i} &= \frac{\partial V_j}{\partial \rho_i}\frac{\partial K}{\partial V_j} \\ &= -\frac{1}{2}\frac{t_i}{V}.\end{aligned} \tag{4.42}$$

Then, (4.39) becomes

$$\begin{aligned}&T\left\{\exp[\rho_1 - 2\rho_2] - 2\exp[-2\rho_1 + \rho_2]\right\} - \frac{t_1}{2V}W = 0 \\ &T\left\{-2\exp[\rho_1 - 2\rho_2] + \exp[-2\rho_1 + \rho_2]\right\} - \frac{t_2}{2V}W = 0.\end{aligned} \tag{4.43}$$

Since the equations have symmetry (4.34), it makes sense to look for symmetric solutions. That is we only consider the case $\rho_1 = \rho_2 = \rho$. Let $r = \Re\rho$ and $\theta = \Im\rho$. Then, $\frac{t}{2V} = \frac{3}{4r}$ and the above equations become

$$(r + \frac{3}{2})\exp[-r - i\theta] = -\frac{3W_0}{4T} \tag{4.44}$$

The solution to this equation is known:

$$r = -\left(\frac{3}{2}\right) - \mathcal{W}\left(\frac{3W_0}{4T}\exp[-\frac{3}{2} + i\theta]\right) \tag{4.45}$$

where $\mathcal{W}(x)$ is the Lambert W-function. It is the solution to

$$\mathcal{W}(x)\exp[\mathcal{W}(x)] = x. \tag{4.46}$$

$\mathcal{W}(x)$ is real for $x \geq -\frac{1}{e}$ and is double-valued for $-\frac{1}{e} < x < 0$. The two branches are called $\mathcal{W}_0(x)$ and $\mathcal{W}_{-1}(x)$ and satisfy

$$\mathcal{W}_0\left(-\frac{1}{e}\right) = \mathcal{W}_{-1}\left(-\frac{1}{e}\right) = -1, \quad \mathcal{W}_0(0) = 0, \quad \mathcal{W}_{-1}(0) = -\infty. \tag{4.47}$$

From the above properties of the Lambert W-function, we conclude that if $\frac{W_0}{T}e^{i\theta}$ is real and negative(but, not too negative), r is real. We can always make $\frac{W_0}{T}e^{i\theta}$ real and negative by tuning the value of θ. Moreover, to get large positive r, we need to take the $\mathcal{W}_{-1}$ branch and $\left|\frac{W_0}{T}\right|$ has to be small. In fact, one of the main principles of KKLT is that stabilizing the complex structure moduli at $|W_0| \ll 1$ should lead to a stabilization of the Kähler moduli at large radius. Here, we see that explicitly, as

$$r = -\left(\frac{3}{2}\right) - \mathcal{W}_{-1}\left(\frac{3W_0}{4T}\exp[-\frac{3}{2} + i\theta]\right) \tag{4.48}$$

goes to $+\infty$ when W_0 goes to 0. This is self-consistent with our supergravtiy approximation.

4.5 Conclusion

In this chapter, we reviewed the F-theory compactification on an elliptically fibered Calabi-Yau, X, and its moduli stabilization. The complex structure moduli are stabilized by turning on the flux and the Kähler structure moduli are stabilized by non-perturbative effects such as instatons and gaugino condensations. The stabilization of all Kähler moduli is not generic and actually, for one parameter model, it is not possible. Grassi[30] classified all fano 3-folds whose Kähler moduli are stabilized by this mechanism and toric examples are considered in [18]. We considered a non-toric example. Our example has a supersymmetric analytic solution and the solution has large volume that is consistent with our supergravity approximation.

Since our example is non-toric, we did not consider the complex moduli stabilization. It would be interesting to find an example where one can solve both Kähler and complex structure moduli stabilization analytically.

Also, in the literature, the complex structure moduli stabiliztion is worked out only in the orientifold limit. The orientifold limit is rather special locus in the moduli space of the F-theory compactification. In this limit, the string coupling g_s is zero. Since the complex structure is determined from the superpotential, it is desirable not to assume the orientifold limit, work with the full F-theory superpotential (4.11) and compare the results with the orientifold limit.

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